UTILITY COPULAS

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ABSTRACT. This paper introduces the concept of the utility copula, a function which incorporates the dependence information between (or among) variables of a utility function. A utility copula is a natural extension of the ordinary (probability) copula and the Lévy copula, but relates to a different domain, typically bounded from below in its variables. Thus the utility measure does not go to zero at minus infinity, as in the case of a probability measure, nor to zero at plus infinity, as in the case of a Lévy measure. This qualification requires the non-trivial implementation of the Fundamental Theorem of the Calculus for valuation of the utility copula. Accordingly, issues arise concerning the equivalence of restricted *vs.* marginal utility functions, leading to recognition of the distinction between *regular* and *irregular* utility functions, defined within. The development proceeds to examples of utility copulas, and further to the construction of bivariate (or multivariate) utility functions from a utility copula and marginal utility functions. Further, the paper presents specific tests of necessity and sufficiency on candidate utility copulas, drawn from the spaces of ordinary and Lévy copulas, and provided with marginal utility functions, to assure that a constructed bivariate function be a utility function. Suggestions for applications, and conclusions, follow.

PROLOGUE

„Endlich kann kein Ding Wert sein, ohne Gebrauchsgegenstand zu sein. Ist es nutzlos, so ist auch die in ihm enthaltene Arbeit nutzlos, zählt nicht als Arbeit und bildet daher keinen Wert.“

“Lastly nothing can have value, without being an object of utility. If the thing is useless, so is the labor contained in it; the labor does not count as labor, and therefore creates no value.”

— Karl Marx, *Das Kapital*, 1867 [Part 1, Chapter 1, Section 1, *at end*]

1. INTRODUCTION

Economists for generations have spoken of pairs of commodities and their relationships. Among such pairs are these.

- Wheat and Oats
- Guns and Butter
- Mean and Variance

The pairs have some similarities and some differences.

- Wheat and Oats are substitutes.
  Both are nutritious grains.

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The author wishes to thank the many who have gone before to develop utility theory to the honored place it enjoys in the history of economic thought.

- Guns and Butter are complements.
  One cannot eat guns or shoot butter.
- One prefers higher values of the Mean,
  but lower values of the Variance.

This study considers utility theory relating to pairs as these, and in particular to the application of copula theory to assist in understanding the dependence relationships. The theory in recent years has developed rapidly for application areas as finance and insurance. The two primary realms of this theory are ordinary (probability) copulas which consider random variables, and Lévy copulas, which consider stochastic processes with jumps. Absent from the literature has been application of copula theory to utility functions, a seemingly obvious field for inquiry. Onto this field humbly steps the author, with *homage* to the established works.

For general background reading on probability and Lévy copulas, along with their applications, please look to these references: (Genest and Rivest 1993; Shih and Louis 1995; Nelsen 1998; Cherubini, Luciano, and Vecchiato 2004; Cont and Tankov 2004; Barndorff-Nielsen and Lindner 2007; Kallsen and Tankov 2006; Kettler 2006).

The paper sets forth *seriatim* reflections upon utility function spaces, the utility copula and its construction, two examples to illustrate the ideas, inversion to a bivariate utility function from from marginal utility functions and a chosen utility copula, and a discussion of using ordinary and Lévy copulas for these constructions. Conclusions follow.

2. Utility function spaces

The space of utility functions of one variable U_1 is this.

$$U_1 := \left\{ T(x) \in C^2 : [a,\infty) \to \mathbb{R} \;\middle|\; t(x) := T'(x) > 0,\ t'(x) = T''(x) < 0 \right\}$$

The space of utility functions of two variables U_2 is this.

$$U_2 := \left\{ T(x,y) \in C^2 : [a,\infty) \times [b,\infty) \to \mathbb{R} \;\middle|\; \frac{\partial T}{\partial x} > 0,\ \frac{\partial T}{\partial y} > 0;\ \frac{\partial^2 T}{\partial x^2} < 0,\ \frac{\partial^2 T}{\partial x^2} < 0; \right. \\ \left. \frac{\partial^2 T}{\partial x \partial y} = \frac{\partial^2 T}{\partial y \partial x} < 0;\ \frac{\partial^2 T}{\partial x^2}\frac{\partial^2 T}{\partial y^2} - \frac{\partial^2 T}{\partial x \partial y}\frac{\partial^2 T}{\partial y \partial x} > 0 \right\} \tag{2.1}$$

The last of these conditions is that the determinant of the Hessian matrix of T be positive, a condition which ensures that the the utility function be convex.

3. Utility copula

3.1. Utility function, utility measure and the Fundamental Theorem of the Calculus.

$$\text{Let} \quad T(x,y) : [a,\infty) \times [b,\infty) \to \mathbb{R}$$

$$\begin{aligned} &\text{Consider } A_x := [a,x] \times [b,\infty),\ x \geqslant a \\ &\text{and} \qquad B_y := [b,y] \times [a,\infty),\ y \geqslant b \end{aligned}$$

$$I(x,y) := -\int_{A_x \cap B_y} \frac{\partial^2 T}{\partial x \partial y}\, dx\, dy = -T(x,y) + T(x,b) + T(a,y) - T(a,b) > 0, \tag{3.1}$$

where $-\dfrac{\partial^2 T}{\partial x \partial y}$ is the *measure* of $T(x,y)$. Let the restrictions of $T(x,y)$ to its lower bounds be these.

$$\begin{aligned} T_1(x) &:= T(x,b) \\ T_2(y) &:= T(a,y) \\ \text{Thus}\quad T(a,b) &= T_1(a) = T_2(b) =: k \end{aligned}$$

3.2. **Domain of the utility copula.** Let the point $(u,v) \in \mathbb{R}^2_+$ be as follows.

$$\begin{aligned} u &= \lim_{y\to\infty} I(x,y) = T_1(x) - T_1(a) = \int_a^x \frac{dT_1}{ds}\, ds > 0 \\ v &= \lim_{x\to\infty} I(x,y) = T_2(y) - T_2(b) = \int_b^y \frac{dT_2}{dt}\, dt > 0 \end{aligned} \tag{3.2}$$

by Equation (3.1) if

$$\lim_{y\to\infty} \big(T(x,y) - T(a,y)\big) = \lim_{x\to\infty} \big(T(x,y) - T(x,b)\big) = 0 \tag{3.3}$$

This result is ensured by the assumptions, now made,

$$\lim_{y\to\infty} \frac{\partial T}{\partial x} = \lim_{x\to\infty} \frac{\partial T}{\partial y} = 0 \tag{3.4}$$

This assumption on the limits of the first partial derivatives justifies calling the restriction of $T(x,y)$ to the boundaries $y = b$ and $x = a$ the *marginal utility functions* $T_1(x)$ and $T_2(y)$. That the conditions of Equations (3.4) are also necessary follows indirectly from the assumption that either first derivative is bounded above zero. Assuming γ is such a bound, then for some point x or y one has,

$$\begin{aligned} T(x,y) - T(a,y) &> (x-a)\gamma > 0 \\ \text{or}\quad T(x,y) - T(x,b) &> (y-b)\gamma > 0\,, \end{aligned}$$

independent, respectively, of y or x insofar as the first derivatives are monotone decreasing. As the same bounds apply in the limits, u and v cannot be the restrictions of $T(x,y)$ to $y = b$ and $x = a$, as assumed.

These observations inspire the following definition.

Definition 3.1. A bivariate (or multivariate) utility function for which its marginal utility functions equal the restricted function to the boundaries is deemed *regular*. All others are *irregular*.

This paper concentrates on regular utility functions. A parallel theory for irregular functions appears possible, but is deferred. An example of an irregular utility function is the Cobb-Douglas, $T_{CD}(x,y) = x^\alpha y^\beta$, for positive exponents, because the first partial derivatives are unbounded at infinity.

4. Construction of the utility copula

Referring again to Equation (3.1), making substitutions for the marginal functions and the constant k, one has

$$\begin{aligned} C(u,v) &:= C\big(T_1(x)-k, T_2(y)-k\big) = I(x,y) \\ &= -T(x,y) + T_1(x) + T_2(y) - k \\ &= -T\big(T_1^{-1}(u+k), T_2^{-1}(v+k)\big) + (u+k) + (v+k) - k \end{aligned}$$

$$C(u,v) = -T\big(T_1^{-1}(u+k), T_2^{-1}(v+k)\big) + (u+v) + k > 0 \tag{4.1}$$

4.1. **Boundary conditions.** The utility copula as constructed adheres both to the ground condition and the uniform margin condition.

The ground condition …

$$C(u,0) = C(0,v) = 0$$

The uniform margin condition …

$$C_1(u) = C(u,\infty) = u, \text{ and } C_2(v) = C(\infty,v) = v$$

4.2. **Utility copula measure.** The utility copula has a measure computed from the utility function measure by a straightforward application of the chain rule.

$$c(u,v) := \frac{\partial^2 C}{\partial u \partial v} = -\frac{\dfrac{\partial^2 T}{\partial x \partial y}}{\dfrac{dT_1}{dx}\dfrac{dT_2}{dy}} \tag{4.2}$$

5. Examples

Example 5.1. *A logarithmic substitution utility function —*

$T(x,y) = \log(x+y-1)$, on $[1,\infty)^2$. The margins are

$$T_1(x) = \log x$$
$$T_2(y) = \log y$$

In this example $k = 0$, and the utility copula

$$C(u,v) = u + v - \log(e^u + e^v - 1), \quad \text{on } [0,\infty)^2 \tag{5.1}$$

by Equation (4.1). See Figure 1.

One may extend this copula to a one-parameter family.

$$C_\theta(u,v) = \left[u^\theta + v^\theta - \log\Big(\exp\left(u^\theta\right) + \exp\left(v^\theta\right) - 1\Big)\right]^{\frac{1}{\theta}},$$

on $[0,\infty)^2$, where $\theta \in (0,\infty)$.

Example 5.2. *A Gaussian utility function —*

$T(x,y) = (2\pi)^{-1} - g(x,y)$, on $[1,\infty)^2$, where

$$g(x,y) := \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)$$

$$\text{Then}\quad T_1(x) = \frac{1}{2\pi} - g(x,1) = \frac{1}{2\pi} - f(x)f(1)$$

$$\text{and}\quad T_2(y) = \frac{1}{2\pi} - g(1,y) = \frac{1}{2\pi} - f(1)f(y),$$

$$\text{where}\quad f(x) := \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

The copula $C(u,v)$ develops like this.

$$u = T_1(x) - T_1(1) = g(1,1) - g(x,1) = f(1)\big(f(1) - f(x)\big)$$
$$v = T_2(y) - T_2(1) = g(1,1) - g(1,y) = f(1)\big(f(1) - f(y)\big)$$

In this example $k = (2\pi)^{-1} - g(1,1)$, and the utility copula

$$C(u,v) = \frac{uv}{g(1,1)} = 2\pi e \cdot uv, \quad \text{on } [0,(2\pi e)^{-1}]^2 = [0,g(1,1)]^2 \tag{5.2}$$

by Equation (4.1). This is an independent copula, discussed below. See Figure 2.

Note that the domain of this copula is $[0,g(1,1)]^2$, at which outer bounds the margins are uniform. The finite domain owes to the fact that the utility function is bounded. In general the copular bounds are these, which may be $+\infty$.

$$\begin{aligned}\alpha &:= \liminf_{x\to\infty}\big(T_1(x) - T_1(a)\big)\\ \beta &:= \liminf_{y\to\infty}\big(T_2(y) - T_2(b)\big)\end{aligned} \tag{5.3}$$

Further, to preserve uniformity in the margins of the copula

$$C(\alpha,\beta) = \alpha = \beta$$

One may extend this copula to a one-parameter family in many ways. Here are two.

$$C_\theta^{(1)}(u,v) = \frac{uv}{g(1,1)\big(1 - \theta(1-u)(1-v)\big)}$$

$$C_\theta^{(2)}(u,v) = \frac{uv}{g(1,1)\big(1 - (1-u^\theta)(1-v^\theta)\big)^{\frac{1}{\theta}}}$$

on $[0,g(1,1)]^2$, where $\theta \in [0,1]$, and $C_0^{(2)} := uv/g(1,1)$.

6. Inversion from chosen utility margins

Equation (4.1) provides the means to invert the process of finding a bivariate utility function by starting with a utility copula, choosing margins, and creating a bivariate utility function. Making the necessary substitutions,

$$T(x,y) + T(a,b) = T_1(x) + T_2(y) - C\big(T_1(x) - T_1(a), T_2(y) - T_2(b)\big) \tag{6.1}$$

Note, however, that at this point the function $T(x,y)$ is simply a trial utility function. It is not demonstrated, yet, that such a function with arbitrarily chosen margins is itself a bivariate utility function. It is necessary at the outset, however, to choose margins such that $T_1(a) = T_2(b)$, because that common value becomes $T(a,b)$ in the generated utility function, recognizing the conditions imposed by Equations (5.3).

Questions of sufficiency for a choice of margins and copula to produce a bivariate utility function are addressed more generally in Section 9.

7. Copula types

7.1. **Independence.** The copula of the Gaussian utility function, as in Equation (4.1), closely resembles the independent probability copula, and therefore inspires the thought of the class of utility functions having a copula of the form

$$C(u,v) = \frac{1}{\alpha}uv \tag{7.1}$$

Such a utility function $T(x,y)$ has this measure, by the inversion formula, Equation (6.1).

$$\frac{\partial^2 T}{\partial x \partial y} = \frac{1}{\alpha}\frac{dT_1}{dx}\frac{dT_2}{dy} \tag{7.2}$$

Conversely, any bounded utility function satisfying Equation (7.2) has the copula of Equation (7.1). Application of the Fundamental Theorem of the Calculus reveals that the point

$$(u,v) = \big(T_1(x) - T_1(a), T_2(y) - T_2(b)\big)$$

has the value

$$C(u,v) = \frac{1}{\alpha}\big(T_1(x) - T_1(a)\big)\big(T_2(y) - T_2(b)\big) = \frac{1}{\alpha}uv$$

Definition 7.1. A utility copula $C(u,v) = (1/\alpha)uv$ is an *independent copula*

(1) of density $1/\alpha$, or
(2) of [linear] size α, or
(3) of measure α.

7.2. **Fréchet-Hoeffding upper and lower limit copulas.** These are as follows, respectively. Note that the Fréchet-Hoeffding lower limit copula is valid only in two dimensions, because a negative association cannot be maintained pairwise for more that two variables.

$$C_\uparrow(u,v) = \min(u,v)$$
$$C_\downarrow(u,v) = \max(u+v-\alpha, 0)$$

7.3. **Complementarity.** All complementary utility functions, *e.g.*, Guns and Butter, have separable variables (making the mixed second partial derivative zero.) By extending the definition of the independent copula to unbounded utility functions by letting $\alpha \to \infty$, an unbounded complementary utility function becomes independent. (This is the copula which is zero everywhere, save for the lines at infinity, on which is it uniform.)

A bounded complementary utility function cannot be independent, for it violates Equation (7.2). In fact it cannot even have a copula, for by the definition the margins would be zero, violating the ground condition. This is an uninteresting case.

7.4. **Mean – Variance.** The only concern is the monotone-decreasing desirability of the variance. A suggestion is simply to transform the variance into another variable which is monotone increasing, like its reciprocal. (As an aside, the reciprocal is frequently misidentified as the 'inverse.' Insofar as the variance is a function of a random variable, its inverse must be a random variable, which clearly is not unique.) One then proceeds as in other cases with monotone increasing variables, recovering the variance at a later step, if desired.

8. Construction of utility functions from copulas and margins

8.1. Examples.

Example 8.1. *An unbounded utility function — power margins with a logarithmic substitution copula*

Let

$$T_1(x) = \sqrt{x}$$
$$T_2(y) = \sqrt{y}$$

Then by Equations (5.1) and (6.1)

$$T(x, y) = \log\left[\exp\sqrt{x} + \exp\sqrt{y} - 1\right]$$

on $[0, \infty)^2$. See Figure 3.

Example 8.2. *A bounded utility function — exponential margins with an independent copula*

On $[0, \infty)$ let

$$T_1(x) = \frac{1}{2\pi e}\left(1 - e^{-x}\right)$$
$$T_2(y) = \frac{1}{2\pi e}\left(1 - e^{-y}\right)$$

Then $T_1(0) = T_2(0) = 0$, and by Equations (5.2) and (6.1)

$$T(x, y) = \frac{1}{2\pi e}\left(1 - e^{-(x+y)}\right)$$

on $[0, \infty)^2$. See Figure 4.

One easily verifies that $T(x, y) \in U_2$ in each of the examples above, raising the question, "What conditions are necessary and sufficient to impose, for instance on an ordinary (probability) copula, to ensure that the resulting function is a utility function?" This is the subject of the Subsection to follow, wherein we continue with the theme introduced in Section 6 above.

9. Using copulas from other realms

A natural question is, "Can one use copulas from other realms, such as ordinary (probability) copulas or Lévy copulas as utility copulas?" The answer is, "Yes," subject to some qualifications. First let us address the former, then the latter.

The important distinction between these classes is that one is defined on a bounded domain — the unit square in two dimensions — whereas the other is defined on the first quadrant. This distinction leads us to consider ordinary copulas as candidates for utility copulas when starting with bounded utility function margins, and to consider Lévy copulas as candidates when taking unbounded margins.

9.1. **Ordinary (probability) copulas as utility copulas.** To begin, let $\mathcal{C}(u, v)$ be an ordinary copula, and consider $T_1(x)$ and $T_2(y)$ as utility function margins. One must scale $\mathcal{C}(u, v)$ in order to try it as a utility copula, for the [arbitrary] margins chosen could have limits conforming to Equations (5.3). The evident scaling to a new trial copula $\widehat{\mathcal{C}}(\hat{u}, \hat{v})$ is as follows. (This scaling must be linear so to preserve uniform margins on the new copula — an easy exercise, omitted.)

$$\widehat{\mathcal{C}}(\hat{u}, \hat{v}) = \alpha \mathcal{C}\left(\frac{1}{\alpha}\hat{u}, \frac{1}{\alpha}\hat{v}\right) = \alpha \mathcal{C}(u, v)$$

Incidentally, given this scaling, any derivatives of $\widehat{\mathcal{C}}(\hat{u}, \hat{v})$ are the same as those of $\mathcal{C}(u, v)$, when evaluated at the corresponding points. To wit,

$$\begin{aligned}
\left.\frac{\partial \widehat{\mathcal{C}}}{\partial \hat{u}}\right|_{(\hat{u}, \hat{v})} &= \alpha \left.\frac{\partial \mathcal{C}}{\partial u}\right|_{\left(\frac{1}{\alpha}\hat{u}, \frac{1}{\alpha}\hat{v}\right)} \cdot \left.\frac{\partial u}{\partial \hat{u}}\right|_{(\hat{u}, \hat{v})} \\
&= \alpha \left.\frac{\partial \mathcal{C}}{\partial u}\right|_{\left(\frac{1}{\alpha}\hat{u}, \frac{1}{\alpha}\hat{v}\right)} \cdot \frac{1}{\alpha} \\
&= \left.\frac{\partial \mathcal{C}}{\partial u}\right|_{\left(\frac{1}{\alpha}\hat{u}, \frac{1}{\alpha}\hat{v}\right)},
\end{aligned}$$

and so, *mutatis mutandis*, for the derivatives with respect to $\hat{v}$ and v. This feature makes any test on the derivatives of $\widehat{\mathcal{C}}(\hat{u}, \hat{v})$ transferable *pro forma* to the derivatives of $\mathcal{C}(u, v)$. Drop now these 'hats' from $\widehat{\mathcal{C}}(\hat{u}, \hat{v})$ to simplify.

It remains to verify the requirements of Equation (2.1), while recording any necessary conditions. Let the newly formed test bivariate utility function conform to Equation (6.1). The first requirement is that the first derivatives of $T(x, y)$ be positive. So impose

$$\frac{\partial T}{\partial x} = \frac{dT_1}{dx} - \frac{\partial \mathcal{C}}{\partial u}\frac{dT_1}{dx} = \frac{dT_1}{dx}\left(1 - \frac{\partial \mathcal{C}}{\partial u}\right) > 0 \tag{9.1}$$

Given that $T_1(x) > 0$, this inequality holds if and only if the second factor is positive, that is, if and only if

$$\frac{\partial \mathcal{C}}{\partial u} < 1,$$

and so for the corresponding condition for the partial derivative with respect to v. As noted, these conditions transfer directly to the derivatives on the originally chosen copula, the one defined on the unit square.

Observe, now, that this condition holds for all copulas, for the grounded property $\mathcal{C}(u, 0) = 0$ and the uniform margin property $\mathcal{C}(u, 1) = u$, along with the non-negative measure property

$$\frac{\partial^2 \mathcal{C}}{\partial u \partial v} = \frac{\partial}{\partial v}\frac{\partial \mathcal{C}}{\partial u} > 0, \quad \text{ensure that} \quad \frac{\partial \mathcal{C}}{\partial u} < 1, \quad \text{for otherwise} \quad \left.\frac{\partial}{\partial v}\frac{\partial \mathcal{C}}{\partial u}\right|_{(u, \hat{v})} < 0,$$

for some $\hat{v}$, $v < \hat{v} \leqslant 1$ (by the Mean Value Theorem,) with a similar finding for the partial derivative with respect to v. Therefore these first order constraints are not binding on the choice of copula.

Next, approach the second derivatives of $T(x, y)$, which must be negative. So, impose

$$
\begin{aligned}
\frac{\partial^2 T}{\partial x^2} &= \frac{d^2 T_1}{dx^2}\left(1 - \frac{\partial \mathcal{C}}{\partial u}\right) + \frac{dT_1}{dx}\left(-\frac{\partial}{\partial x}\frac{\partial \mathcal{C}}{\partial u}\right) < 0 \\
\text{or} \quad \frac{\partial^2 T}{\partial x^2} &= \frac{d^2 T_1}{dx^2}\left(1 - \frac{\partial \mathcal{C}}{\partial u}\right) - \frac{dT_1}{dx}\frac{\partial}{\partial x}\frac{\partial \mathcal{C}}{\partial u} < 0 \\
\text{or} \quad \frac{\partial^2 T}{\partial x^2} &= \frac{d^2 T_1}{dx^2}\left(1 - \frac{\partial \mathcal{C}}{\partial u}\right) - \left(\frac{dT_1}{dx}\right)^2 \frac{\partial^2 \mathcal{C}}{\partial u^2} < 0\,, \qquad (9.2)
\end{aligned}
$$

where the terms and factors are all evaluated at corresponding points, and similarly for the second partial derivative with respect to v. The necessary condition, therefore, with parallel construction on v, is that

$$
\frac{d^2 T_1}{dx^2}\left(1 - \frac{\partial \mathcal{C}}{\partial u}\right) < \left(\frac{dT_1}{dx}\right)^2 \frac{\partial^2 \mathcal{C}}{\partial u^2} \qquad (9.3)
$$

Observe that these conditions depend both on the choice of copula, and on the choice of marginal utility functions.

Next, examine the mixed second partial derivatives of $T(x, y)$, which must be negative by the constraints of Equation (2.1). From Equation (9.1), therefore,

$$
\begin{aligned}
\frac{\partial^2 T}{\partial x \partial y} &= \frac{dT_1}{dx}\frac{\partial}{\partial y}\left(-\frac{\partial \mathcal{C}}{\partial u}\right) \\
&= -\frac{dT_1}{dx}\frac{\partial^2 \mathcal{C}}{\partial u \partial v}\frac{dv}{dy}\,, \\
\text{so} \quad \frac{\partial^2 T}{\partial x \partial y} &= -\frac{dT_1}{dx}\frac{dT_2}{dy}\frac{\partial^2 \mathcal{C}}{\partial u \partial v}\,, \qquad (9.4)
\end{aligned}
$$

by the second of Equations (3.2). That this quantity is negative follows from the known attributes of $T_1(x)$ and $T_2(y)$, and the fact that the mixed second partial derivative on a copula is necessarily positive, to preserve the 2-increasing (positive measure) property. This result, being universal, imposes no additional constraint on the choice of copula or marginal utility functions.

Lastly, it is necessary to inspect the Hessian matrix $H(T)$ of $T(x, y)$ for positive definiteness, ensuring convexity. Referring to Equations (9.2) and (9.4) one has,

$$
\begin{aligned}
|H(T)| = &\left[\frac{d^2 T_1}{dx^2}\left(1 - \frac{\partial \mathcal{C}}{\partial u}\right) - \left(\frac{dT_1}{dx}\right)^2 \frac{\partial^2 \mathcal{C}}{\partial u^2}\right]\left[\frac{d^2 T_2}{dy^2}\left(1 - \frac{\partial \mathcal{C}}{\partial v}\right) - \left(\frac{dT_2}{dy}\right)^2 \frac{\partial^2 \mathcal{C}}{\partial v^2}\right] \\
&- \left(\frac{dT_1}{dx}\frac{dT_2}{dy}\frac{\partial^2 \mathcal{C}}{\partial u \partial v}\right)^2
\end{aligned}
$$

Recast this equation, with the obvious substitutions, as follows.

$$
\begin{aligned}
|H(T)| &= [A - B][C - D] - E^2 \\
&= AC - (AD + BC) + (BD - E^2)
\end{aligned}
$$

Now, $AC > 0$ by the assumption on the first partial derivatives of $\mathcal{C}(u, v)$; $(BD - E^2) > 0$ by the fact that $|H(\mathcal{C})| > 0$. With these positive terms, therefore, the positive definiteness of

$H(T)$ is assured by this final condition, restoring the original notation.

$$\frac{d^2T_1}{dx^2}\left(1-\frac{\partial \mathcal{C}}{\partial u}\right)\cdot\left(\frac{dT_2}{dy}\right)^2\frac{\partial^2\mathcal{C}}{\partial v^2}+\left(\frac{dT_1}{dx}\right)^2\frac{\partial^2\mathcal{C}}{\partial u^2}\cdot\frac{d^2T_2}{dy^2}\left(1-\frac{\partial \mathcal{C}}{\partial v}\right)$$
$$<\frac{d^2T_1}{dx^2}\left(1-\frac{\partial \mathcal{C}}{\partial u}\right)\cdot\frac{d^2T_2}{dy^2}\left(1-\frac{\partial \mathcal{C}}{\partial v}\right)+\left[\left(\frac{dT_1}{dx}\right)^2\frac{\partial^2\mathcal{C}}{\partial u^2}\cdot\left(\frac{dT_2}{dy}\right)^2\frac{\partial^2\mathcal{C}}{\partial v^2}-\left(\frac{dT_1}{dx}\frac{dT_2}{dy}\frac{\partial^2\mathcal{C}}{\partial u\partial v}\right)^2\right] \tag{9.5}$$

To recapitulate, Equations (9.3) and (9.5), along with the cohort of the former in the alternate variables y and v, provide a set of necessary and sufficient conditions for an ordinary copula with choice of marginal utility functions to produce a bivariate utility function. Restrictions on a parametric family of copulas imposed by these criteria typically define a convex set of feasible values in the parameter space.

Example 5.2 conforms to these conditions.

For further examples to generate bivariate utility functions, see the Clayton, Gumbel, and Frank copulas as discussed in (Nelsen 1998).

9.2. **Lévy copulas as utility copulas.** Unlike the case of using ordinary copulas as utility copulas for the bounded margin case, Lévy copulas apply in the unbounded case without scaling. If one were to want a scaled Lévy copula, then that is just another Lévy copula, for these have $[0,\infty)^2$ as their domain.

Let, then, $\mathcal{K}(u,v)$ be a Lévy copula. One wishes to construct a bivariate utility function $T(x,y)$ from this Lévy copula and margins $T_1(x)$ and $T_2(y)$. How can this be done, and what minimal conditions does one impose on these functions in order to assure that the resulting construction conforms to the definition? The answer to this question is to apply the inversion formula of Equation (6.1), with $\mathcal{K}(u,v)$ replacing $\mathcal{C}(u,v)$, and then to apply the same tests of Subsection 9.1.

Example 5.1 conforms to these conditions, considering the developed utility copula as a Lévy copula.

For further examples to generate bivariate utility functions, see the Clayton-Lévy, Gumbel-Lévy, and Complementary Gumbel-Lévy copulas as discussed in (Kettler 2006). The latter two of these are introduced in that paper.

10. Conclusions

The application of copula theory to the study of dependency relationships implicit in utility theory would appear to have value for understanding the connections. With the rich literature on ordinary copulas and developing literature on Lévy copulas ready for direct involvement with utility theory it would seem that dependency relationships heretofore unrecognized may soon emerge. These investigations could show us something new about the way people behave when confronting choices. Beyond that, given the interplay of copulas with stochastic processes, it is only natural to consider how utility copulas may evolve over time, both in the theoretical and empirical senses.

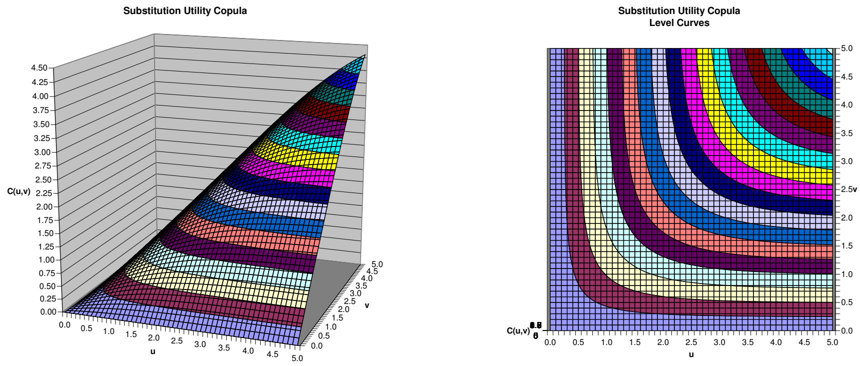

FIGURE 1. Substitution utility copula — perspective and level curves

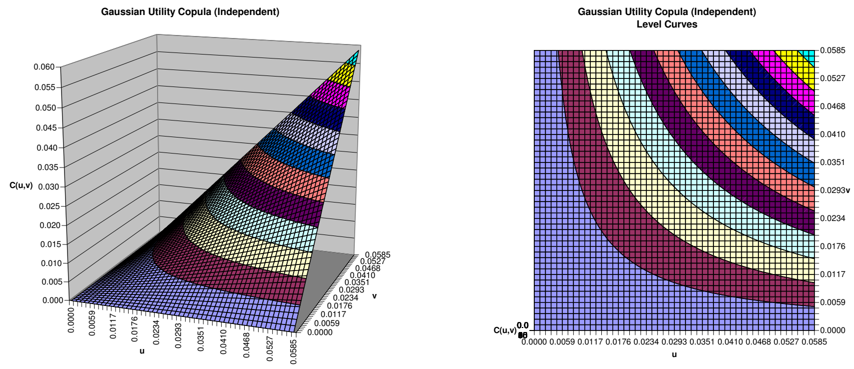

FIGURE 2. Gaussian utility copula (independent) — perspective and level curves

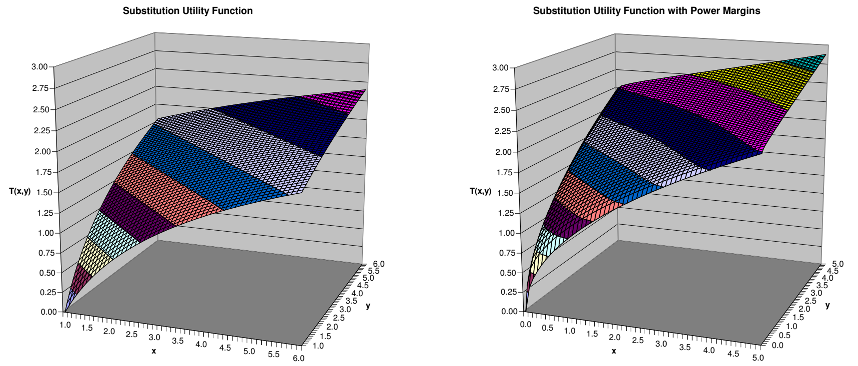

FIGURE 3. Substitution utility function — with logarithm and power margins

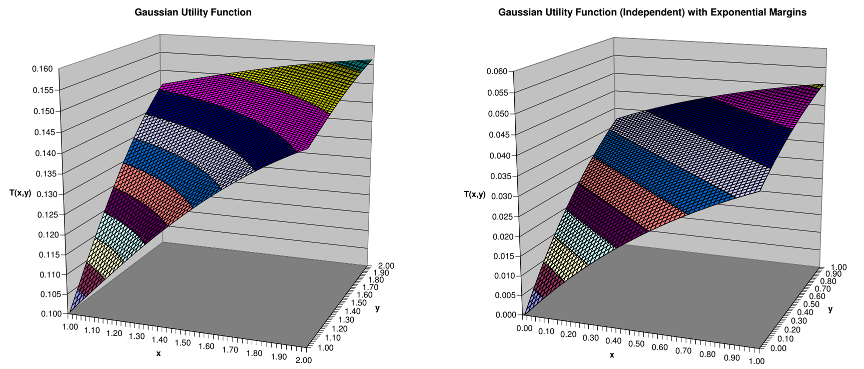

FIGURE 4. Gaussian utility function — with Gaussian and exponential margins

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