# STRATIFICATION OF AN IDEAL AGGREGATE
## WHEN SUBJECT TO THE BRAZIL NUT EFFECT

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ABSTRACT. This paper addresses the levels in an aggregate at which one may find particles of a specified size. Assumed are the size distribution and the allocation of the particles to strata according to the Brazil nut effect, *i.e.*, with the particles arrayed larger above smaller.

## 1. INTRODUCTION

Assume the following. An aggregate in $d$-dimensional Euclidean space $\mathbb{R}^d$ extends in all axial directions indefinitely, except for one, which is called the *height* direction along the positive real axis $\mathbb{R}_+$. The size of particles is a random variable $X$ with known distribution, and the aggregate has finite height $h$. The aggregate fills the space within the interval $[0, h]$, without interstitial space among the particles, as, *e.g.*, occupied by cement in concrete, a constraint later relaxed.

The common sense of this arrangement is that the location of the largest particles of same size is $h$ and the location of the smallest particles of same size is 0. The height $h$ in a natural sense is the number of particles in a set $A \times [0, h] \subset \mathbb{R}^{d-1} \times [0, h]$ multiplied by the average size of the particles. Similarly, the *location* $g(y)$ of particles of size $y$ is the number of particles no larger than $y$ times their average size. The location $g(y)$ is clearly a function of the assumed distribution $F(x)$, $x \in \mathbb{R}_+$, of $X$. An example to entertain is the standard lognormal distribution $\Lambda_{(0,1)}(x) = \Phi_{(0,1)}(\log x)$, the standard normal distribution. Another example is the uniform distribution on the unit interval $U_{[0,1]}(x) = x$.

The following references, and references therein, provide suitable background for this discussion. (Esztermann and Löwen 2004; Rhodes, Takeuchi, Liffman, and Muniandy 2003; Möbius, Lauderdale, Nagel, and Jaeger 2001)

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## 2. Development

To begin, let the location function $g(y)$ be given by

$$g(y) := \int_0^y x \cdot dF(x) = \int_0^y x \cdot f(x)\,dx,$$

if $F(x)$ has a density $f(x)$. This definition conforms to the intuition of the Introduction.

Now, the total mass of particles no larger than $y$ is

$$F(y) = \int_0^y f(x)\,dx,$$

in the event of a density. So the aforementioned average size of particles no larger than $y$, which is the expectation of $X$ on the condition $X \leqslant y$, is

$$A(y) := \frac{\int_0^y x \cdot dF(x)}{F(y)} = \frac{\int_0^y x \cdot f(x)\,dx}{\int_0^y f(x)\,dx},$$

again, in the event of a density. Consequently,

$$g(y) = F(y) \cdot A(y) = \left(\int_0^y f(x)\,dx\right) A(y),$$

in the event of a density, as hypothesized. Consider now an example.

**Example 2.1** (Particle sizes are lognormally distributed)**.** *Let $f(x)$ be the standard lognormal density*

$$\lambda_{(0,1)}(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\log^2 x\right)$$

*Then*

$$g(y) = \frac{1}{\sqrt{2\pi}} \int_0^y \exp\left(-\frac{1}{2}\log^2 x\right) dx$$
$$= \sqrt{e} \cdot \Phi(\log y - 1)$$
$$= \sqrt{e} \cdot \Lambda(y/e)$$

*Now,*

$$h = \lim_{y\to\infty} g(y)$$
$$= \sqrt{e},$$

*which is the mean of the standard lognormal distribution, as anticipated.*

The location function $g(y)$ has two interesting points. One is the inflection point at $y = 1$, giving $g(1) = \sqrt{e} \cdot \Lambda(1/e) \approx 0.2616$, and the other is the location of the average size particle, $g(\sqrt{e}) = \sqrt{e} \cdot \Lambda(1/\sqrt{e}) \approx 0.5087$. Thus the lowest stratum which consists only of particles of size one or smaller occupies $\Lambda(1/e) \approx 0.1587$ of the height $h$, whereas $\Lambda(1/\sqrt{e}) \approx 0.3085$ of the height $h$ contains all the particles of average size or smaller.

**Remark 2.1** (Scaling and interstitial space)**.** *The foregoing discussion assumed standardized distributions for simplicity. It is easy, however, to provide scaling in order to accommodate ideal aggregates of arbitrary height, say* $H$*, and to allow for interstitial space.*

*As for scaling, we can consider two varieties, which we may call the* alias *and* alibi *approaches. With the alias approach one simply redefines the unit of measurement. As in Example 2.1, where the height of the aggregate was* $\sqrt{e}$*, and the unit of measurement was undefined, simply define it a desired, centimeters, say. With the alibi approach define a new random variable, say* $Y = \alpha \cdot X$, $\alpha > 0$*, and reset the distribution from* $F(x)$ *to* $G(x) = F(x/\alpha)$*.*

*As for interstitial space, consider the fractal nature of aggregates in general. The spaces around particles must be proportional to the sizes of the particles, or else it would be possible to tell the degree of magnification. Therefore, allowing for interstitial space one only need choose a scaling factor* $\alpha$*, as above, to accommodate the expansion, say* $\alpha = 1.5$*. Such factors could well be dimension dependent, considering,* e.g.*, the observation that the volume of a hypersphere is less and less compared to the volume of the enclosing hypercube as dimension increases.*

**Example 2.2** (Particle sizes are uniformly distributed)**.** *Let* $f(x)$ *be the standard uniform density*

$$u_{[0,1]}(x) \equiv 1,$$

*Then*

$$g(y) = \int_0^y x\, dx$$
$$= \frac{y^2}{2}$$

*Now,*

$$h = g(1)$$
$$= \frac{1}{2},$$

*because the largest particle is of size one. The height* $h$ *is the mean of the standard uniform distribution, as anticipated.*

Note that the location function has no inflection points, in contrast to the lognormal case of Example 2.1 above. The average size particle is located at $g(0.5) = 0.25$ of the height $h$.

## REFERENCES

Esztermann, A. and H. Löwen (2004, Oct.). Colloidal Brazil-nut effect in sediments of binary charged suspensions. *Europhys. Lett. 68*(1), 120–126.

Möbius, M. E., B. E. Lauderdale, S. R. Nagel, and H. M. Jaeger (2001, Nov.). Brazil-nut effect: Size separation of granular particles. *Nature 414*, 270.

Rhodes, M., S. Takeuchi, K. Liffman, and K. Muniandy (2003). The role of interstitial gas in the Brazil nut effect. *Granular Matter 5*(3), 107–114.

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