

# A STOCHASTIC VARIANCE RATIO TEST TO DISCRIMINATE BETWEEN TIME AND SPACE EFFECTS OF DISCREPANCY BETWEEN FILTRATIONS

PAUL C. KETTLER

ABSTRACT. This study provides three results of note. First, analysis of information can be abstracted from monetary values in special cases to variability of returns in general cases. Second, with an appropriately chosen model one can separate and calculate independently the influences on a stochastic process of having either more timely information, or better quality information, or both. Third, one can bring methods of statistical inference to stochastic analysis. The paper develops a theory of applying variance ratio tests to problems of inference, and by so doing, enables one to determine the separate influences of timely and superior information on results. Numerous examples illustrate the theory.

## 1. INTRODUCTION

The value of information as an abstract concept has been investigated by many. Among those directly relevant to mathematical inquiry are these (Øksendal 2005; Arriojas, Hu, Mohammed, and Pap 2007; Liu, Peleg, and Subrahmanyam 2007) and references therein.

The first of these papers delves into the values of three kinds of processes dependent on a nest of filtrations: a classical case with complete information, a partial observation case, and an insider case. The paper constructs a wealth process in each case, and imposes a logarithmic utility function. The partial observation case employs an adapted process to the classical filtration, whereas the insider case requires a forward stochastic integral development for an anticipative process. Each process allows a value, which is the expectation of the logarithmic terminal wealth. The worth of the information lost or gained in the sub- and super-filtrations, therefore, is stated as the differences of these respective expectations.

The second of these papers investigates the value of information in the ‘constant absolute risk aversion’ (CARA) and ‘constant relative risk aversion’ (CRRA) settings, represented by power and exponential utility functions, respectively. The model presented is more general than that of the former paper, constructed to include consumption and allocation decisions, portfolio weights, stock beta values [of the Capital Asset Pricing Model (CAPM)], systematic risk, and idiosyncratic uncertainty. The model is more intricate, therefore, but consistent with the norm of traditional financial economic studies, is less rigorous. The measure theoretic probability methodology of the compared paper is more so, and thus its conclusions have greater force. Nonetheless, the latter paper forms interesting and believable conclusions on the value and impact of information, reinforced with an included empirical study. One of these conclusions is that information is more valuable to the wealthy agent in the

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CARA setting, but is insensitive to wealth in the CRRA setting. In both cases, information has significant impact on consumption and asset allocation.

The third of these papers studies a non-linear stochastic differential delay equation in a complete market setting, thus enabling a martingale measure equivalent to the assumed. This study focuses on the delay of information specifically in the definition of the problem anticipating as to a Brownian motion adapted to its natural filtration.

Herein I present a variance ratio statistic, in analogy to the  $R^2$  statistic of linear regression, but in reference to comparisons of the filtrations of stochastic process. Two versions appear, an  $R_T^2$  statistic for time-compared processes, and an  $R_S^2$  statistic for space-compared processes<sup>1</sup>. Each represents the fraction of variance of an included filtration ‘explained’ by the including one, and thus is a comparative measure between the two. In the real world of financial information, time decay is usually rapid, whereas spatial relations are more persistent, an observation offered *en passant*.

## 2. A VARIANCE RATIO STATISTIC

Assume filtered probability spaces  $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P\}$  and  $\{\Omega, \mathcal{G}, \{\mathcal{G}_t\}, P\}$ . Consider the following constructions, where  $\sigma^2(\cdot)$  is the variance assumed to exist a.s.  $P$ ,  $\forall t, 0 \leq t \leq T$ ;  $u(\cdot)$  is a utility function and  $J_T$  is the terminal value of a performance functional, which could be wealth, and generally is dependent on several additional variables.

$$(2.1) \quad R_T^2 := 1 - \frac{\sigma^2(E[u(J_T)|\mathcal{F}_t])}{\sigma^2(E[u(J_T)|\mathcal{F}_s])}, \quad 0 \leq s \leq t \leq T$$

$$(2.2) \quad R_S^2 := 1 - \frac{\sigma^2(E[u(J_T)|\mathcal{G}_t])}{\sigma^2(E[u(J_T)|\mathcal{F}_t])}, \quad \mathcal{G}_t \supseteq \mathcal{F}_t$$

$$(2.3) \quad R_{TS}^2 := 1 - \frac{\sigma^2(E[u(J_T)|\mathcal{G}_t])}{\sigma^2(E[u(J_T)|\mathcal{F}_s])}, \quad 0 \leq s \leq t \leq T, \mathcal{G}_t \supseteq \mathcal{F}_s$$

$$R_{\tilde{TS}}^2 := 1 - \frac{\sigma^2(E[u(J_T)|\mathcal{G}_s])}{\sigma^2(E[u(J_T)|\mathcal{F}_t])}, \quad 0 \leq s \leq t \leq T, \mathcal{G}_s \supseteq \mathcal{F}_t$$

The mnemonic significance of the tilde over the ‘time’ index is that the roles of  $s$  and  $t$  are reversed with respect to the formulation of Equation (2.3).

$$R_{\tilde{TS}}^2 := 1 - \frac{\sigma^2(E[u(J_T)|\mathcal{F}_t])}{\sigma^2(E[u(J_T)|\mathcal{G}_s])}, \quad 0 \leq s \leq t \leq T, \mathcal{F}_t \supseteq \mathcal{G}_s$$

The mnemonic significance of the tilde over the ‘space’ index is that the roles of  $\mathcal{F}_\bullet$  and  $\mathcal{G}_\bullet$  are reversed with respect to the formulation of Equation (2.3).

**Example 2.1** (Brownian motion). *Let  $u(J_t)$  be a Brownian motion, or equivalently, the logarithmic utility of a geometric Brownian motion. Let the lesser filtration  $\mathcal{F}_s$  be given by an information delay*

<sup>1</sup>Observe that the subscript initials are set in roman, as they are mnemonic devices, not variables.

from time  $s$  to time  $t$ . Insofar a variance grows linearly with time, Equation (2.1) becomes

$$R_T^2 = 1 - \frac{T-t}{T-s}$$

$$R_T^2 = \frac{t-s}{T-s}$$

**Example 2.2** (Compound Poisson process). Let  $u(J_t)$  be a compound Poisson process, with intensity  $\lambda$  and i.i.d. variables  $\{D_i = D\}$ . The variance of the process is  $\lambda t E[D^2]$ . So, Equation (2.1) becomes

$$(2.4) \quad R_T^2 = 1 - \frac{T - \lambda t E[D^2]}{T - \lambda s E[D^2]}$$

$$R_T^2 = \frac{\lambda t E[D^2] - \lambda s E[D^2]}{T - \lambda s E[D^2]}$$

$$R_T^2 = \frac{(t-s)\lambda E[D^2]}{T - \lambda s E[D^2]}$$

$$R_T^2 = \frac{(t-s)\lambda}{T/E[D^2] - \lambda s}$$

As a special case consider the Normal Inverse Gaussian distribution  $\text{NIG}(\mu, \alpha, \beta, \delta)$ , which has mean  $\mu + \delta\beta/\gamma$  and variance  $\delta\alpha^2/\gamma^3$ , given  $\gamma := \sqrt{\alpha^2 - \beta^2}$ . Let the mean be zero to produce a martingale. Then  $E[D^2]$  equals the variance, and Equation (2.4) becomes

$$R_T^2 = \frac{(t-s)\lambda}{(\gamma^3/\delta\alpha^2)T - \lambda s}$$

**Example 2.3** (Constant variance ratio process). Let

$$\sigma^2(E[u(J_T)|\mathcal{G}_t]) = \rho^2 \cdot \sigma^2(E[u(J_T)|\mathcal{F}_t]), \quad 0 \leq \rho^2 \leq 1$$

Then Equation (2.2) becomes

$$R_S^2 = 1 - \rho^2$$

**Example 2.4** (Linearly diminishing variance ratio process). Let

$$\sigma^2(E[u(J_T)|\mathcal{G}_t]) = ((T-t)/T)\rho^2 \cdot \sigma^2(E[u(J_T)|\mathcal{F}_t]), \quad 0 \leq \rho^2 \leq 1$$

Then Equation (2.2) becomes

$$R_S^2 = 1 - \frac{T-t}{T}\rho^2$$

$$R_S^2 = (1 - \rho^2) + \frac{t}{T}\rho^2$$

**Example 2.5** (Brownian motion – constant variance ratio process).

$$R_{TS}^2 = 1 - \frac{T-t}{T-s}\rho^2, \quad 0 \leq \rho^2 \leq 1$$

**Example 2.6** (Compound Poisson – linearly diminishing variance ratio process).

$$R_{TS}^2 = 1 - \frac{T - \lambda t E[D^2]}{T - \lambda s E[D^2]} \cdot \frac{T - t}{T} \rho^2, \quad 0 \leq \rho^2 \leq 1$$

As a special case consider the Normal Inverse Gaussian distribution  $\text{NIG}(\mu, \alpha, \beta, \delta)$ .

$$R_{TS}^2 = 1 - \frac{T - \lambda t \cdot \delta \alpha^2 / \gamma^3}{T - \lambda s \cdot \delta \alpha^2 / \gamma^3} \cdot \frac{T - t}{T} \rho^2, \quad 0 \leq \rho^2 \leq 1$$

**Example 2.7** (Brownian motion – constant variance ratio process, time reversed).

$$R_{TS}^2 = 1 - \frac{T - s}{T - t} \rho^2, \quad 0 \leq \rho^2 \leq \frac{T - t}{T - s}$$

**Example 2.8** (Compound Poisson – linearly diminishing variance ratio process, space reversed).

$$R_{TS}^2 = 1 - \frac{T - \lambda t E[D^2]}{T - \lambda s E[D^2]} \cdot \frac{T - t}{T \rho^2}, \quad \frac{T - \lambda t E[D^2]}{T - \lambda s E[D^2]} \cdot \frac{T - t}{T} \leq \rho^2 < \infty$$

As a special case consider the Normal Inverse Gaussian distribution  $\text{NIG}(\mu, \alpha, \beta, \delta)$ .

$$R_{TS}^2 = 1 - \frac{T - \lambda t \cdot \delta \alpha^2 / \gamma^3}{T - \lambda s \cdot \delta \alpha^2 / \gamma^3} \cdot \frac{T - t}{T \rho^2}, \quad \frac{T - \lambda t \cdot \delta \alpha^2 / \gamma^3}{T - \lambda s \cdot \delta \alpha^2 / \gamma^3} \cdot \frac{T - t}{T} \leq \rho^2 < \infty$$

### 3. MODEL-DIRECTED F-RATIO TESTS FOR EQUALITY OF FILTRATIONS

If one chooses a model, for instance one of those in the examples of Section 2, then an examination of data gathered pursuant to the model provides parameter estimates by one of several methods. Maximum likelihood [ML] comes immediately to mind, as do other procedures, such as ordinary least squares [OLS]. With parameters selected from one data set, viewed as a control set, then one may take another, independent, sample, generating statistics from the model, in particular the various  $R_{\bullet}^2$  statistics described.

A simple F-ratio test for equality of variance is not useful in these cases, for the tacit assumption is that variances are not equal, except for the [uninteresting] cases of equality of filtrations. However, that said, a reduced F-ratio test is feasible, constructed by comparing the variance with respect to the smaller filtration to the variance of the larger filtration reduced by the factor  $1 - R_{\bullet}^2$ . In this regard, recast Equations (2.1) and (2.2) as follows.

$$\frac{\sigma^2(E[u(J_T)|\mathcal{F}_t])}{(1 - R_T^2)\sigma^2(E[u(J_T)|\mathcal{F}_s])} \propto 1, \quad 0 \leq s \leq t \leq T$$

$$\frac{\sigma^2(E[u(J_T)|\mathcal{G}_t])}{(1 - R_S^2)\sigma^2(E[u(J_T)|\mathcal{F}_t])} \propto 1, \quad 0 \leq t \leq T$$

Restating the examples of Section 2 gives this new set, which serve as direct models for applying the F-ratio test for fitness.

**Example 3.1** (Brownian motion).

$$\frac{t - s}{R_T^2[T - s]} \propto 1$$

**Example 3.2** (Compound Poisson process).

$$\frac{(t-s)\lambda}{R_T^2 [T/E[D^2] - \lambda s]} \propto 1$$

And for the special case of the NIG process —

$$\frac{(t-s)\lambda}{R_T^2 [(\gamma^3/\delta\alpha^2)T - \lambda s]} \propto 1$$

**Example 3.3** (Constant variance ratio process).

$$\frac{1 - \rho^2}{R_\xi^2} \propto 1, \quad 0 \leq \rho^2 \leq 1$$

**Example 3.4** (Linearly diminishing variance ratio process).

$$\frac{(1 - \rho^2) + \frac{t}{T}\rho^2}{R_\xi^2} \propto 1, \quad 0 \leq \rho^2 \leq 1$$

**Example 3.5** (Brownian motion – constant variance ratio process).

$$\frac{T-t}{T-s}\rho^2 \propto 1, \quad 0 \leq \rho^2 \leq \frac{T-s}{T-t}$$

**Example 3.6** (Compound Poisson – linearly diminishing variance ratio process).

$$(3.1) \quad \frac{T - \lambda t E[D^2]}{T - \lambda s E[D^2]} \cdot \frac{T-t}{T} \rho^2 \propto 1, \quad \frac{T - \lambda s E[D^2]}{T - \lambda t E[D^2]} \cdot \frac{T}{T-t} \leq \rho^2 < \infty$$

As a special case consider the Normal Inverse Gaussian distribution  $\text{NIG}(\mu, \alpha, \beta, \delta)$ .

$$\frac{T - \lambda t \cdot \delta\alpha^2/\gamma^3}{T - \lambda s \cdot \delta\alpha^2/\gamma^3} \cdot \frac{T-t}{T} \rho^2 \propto 1, \quad \frac{T - \lambda s \cdot \delta\alpha^2/\gamma^3}{T - \lambda t \cdot \delta\alpha^2/\gamma^3} \cdot \frac{T}{T-t} \leq \rho^2 < \infty$$

**Example 3.7** (Brownian motion – constant variance ratio process, time reversed).

$$\frac{T-s}{T-t}\rho^2 \propto 1, \quad 0 \leq \rho^2 \leq \frac{T-t}{T-s}$$

**Example 3.8** (Compound Poisson – linearly diminishing variance ratio process, space reversed).

$$\frac{T - \lambda t E[D^2]}{T - \lambda s E[D^2]} \cdot \frac{T-t}{T\rho^2} \propto 1, \quad \frac{T - \lambda t E[D^2]}{T - \lambda s E[D^2]} \cdot \frac{T-t}{T} \leq \rho^2 < \infty$$

As a special case consider the Normal Inverse Gaussian distribution  $\text{NIG}(\mu, \alpha, \beta, \delta)$ .

$$\frac{T - \lambda t \cdot \delta\alpha^2/\gamma^3}{T - \lambda s \cdot \delta\alpha^2/\gamma^3} \cdot \frac{T-t}{T\rho^2} \propto 1, \quad \frac{T - \lambda t \cdot \delta\alpha^2/\gamma^3}{T - \lambda s \cdot \delta\alpha^2/\gamma^3} \cdot \frac{T-t}{T} \leq \rho^2 < \infty$$

## 4. INSIDER TRADES

The way is clear now to specify tests for the purpose of deciding whether or not certain trading patterns reveal the use of inside information. The superior information can come in the form of a time advantage or a space advantage, or both. In the former instance information is acquired by the insider before it is available to the general trading public; in the latter case the information is of premium quality. The usual case in actual market circumstances is that the incremental information is both better and earlier than that available to the lesser informed populace. In practice, the time advantage is evanescent, whereas the space advantage is persistent.

Take, for illustration, Equation (3.1) of Example 3.6, repeated below, the variational expression for the compound Poisson – linearly diminishing variance ratio process. Assume that all quantities are known, except for  $s$  and  $\rho^2$ . Determining these parameters establishes the effects, respectively, of the time and space discrepancies in information. Variances of samples of returns from trading patterns from each realm — superior and inferior information — serve to define implicitly the relationship between the parameters. Let, therefore,

$$(4.1) \quad \varphi(s, \rho^2) := \frac{T - \lambda t \cdot \delta \alpha^2 / \gamma^3}{T - \lambda s \cdot \delta \alpha^2 / \gamma^3} \cdot \frac{T - t}{T} \rho^2 \propto 1, \quad \frac{T - \lambda s \cdot \delta \alpha^2 / \gamma^3}{T - \lambda t \cdot \delta \alpha^2 / \gamma^3} \cdot \frac{T}{T - t} \leq \rho^2 < \infty$$

The first factor serves to reduce the expected variation in the returns of the lesser informed trader to that of the insider *vis à vis* the time differential, whereas the second factor serves the same purpose *vis à vis* the space differential. The question arises, “How does one choose a pair?” Insofar as an increased discrepancy of information can be explained by lower values in either  $s$  or  $\rho^2$ , it is correct to assume an inverse relationship between them. The examples of this section including both variables, in particular Equation (4.1), confirm this view.

One choice is to take a Bayesian ‘diffuse prior’ approach, and set each factor of  $\varphi(s, \rho^2)$  to 1. For this discrete case the choice also provides maximal entropy. Then,

$$s = t \\ \rho^2 = \frac{T}{T - t}$$

Thus, in this case the entire insider advantage is in the quality of information, not in the timeliness of its arrival. In the complementary case,  $s = 0$ , providing maximal time advantage, leaving

$$\rho^2 = \frac{T^2}{(T - \lambda t \cdot \delta \alpha^2 / \gamma^3)(T - t)}$$

Another choice involves evaluating the incremental information to an insider acquired from each source — earlier acquisition (smaller value of  $s$ ) or improved quality (smaller value of  $\rho^2$ ), while assuming that the marginal costs for equal value are the same in equilibrium. The idea is to minimize an expression of the form

$$as + b\rho^2$$

subject to a constraint such as

$$(4.2) \quad s\rho^2 = k,$$

where  $a > 0$ ,  $b > 0$ , and  $k > 0$  are constant coefficients. This optimization problem lends itself to a Lagrange multiplier approach, wherein the Lagrangian is

$$\Lambda(s, \rho^2; \lambda) = as + b\rho^2 - \lambda(s\rho^2 - k)$$

A solution is provided by solving the system

$$\begin{aligned} \frac{\partial \Lambda}{\partial s} &= a - \lambda\rho^2 = 0 \\ \frac{\partial \Lambda}{\partial \rho^2} &= b - \lambda s = 0, \end{aligned}$$

along with the equation of constraint, Equation (4.2). This solution is

$$\begin{aligned} (s, \rho^2) &= \left( \sqrt{\frac{bk}{a}}, \sqrt{\frac{ak}{b}} \right) \\ \lambda &= \sqrt{\frac{ab}{k}} \end{aligned}$$

An interpretation of the Lagrange multiplier  $\lambda$  is that of the marginal cost of the combined time and space information at the exchange rate implied by Equation (4.2) at the optimum.

## 5. SAMPLE AND TEST PROCEDURES

Consider any of the generic  $R^2$  statistics of the previous section. The value  $1 - R^2$  is a variance ratio, dividing in each instance the variance for the lesser filtration by the variance for the greater filtration. If this value is estimated by sampling respectively  $(n_1, n_2)$  points from the joint distribution, then Fisher's  $\mathcal{F}$ -statistic is the ratio of variances calculated for  $(\nu_1, \nu_2) = (n_1 - 1, n_2 - 1)$  [numerator and denominator] degrees of freedom. For example, if  $(\nu_1, \nu_2) = (10, 20)$ , then at the 5% significance level one has

$$\mathcal{F}_{.05}(\nu_1, \nu_2) = \mathcal{F}_{.05}(10, 20) = 2.348$$

Inverting this relationship implies that for an  $\mathcal{F}$ -ratio of 2.348, the p-value is .05. For an  $\mathcal{F}$ -ratio of 3.000 the p-value drops to .017510 with the same degrees of freedom. Tables and on-line calculators are readily available for such calculations. The values herein were gleaned from (Soper 2009).

It is well known that the ordinary  $\mathcal{F}$ -ratio test, as described, becomes unreliable as the distributions depart from normal. However, one can take this matter into account by devising p-value intervals, as done with the Durbin – Watson test, *e.g.*, for auto-correlation of regression residuals. This development is outside the scope of the present tract, but is suggested for further research, including the preparation of appropriate tables and on-line calculators.

## 6. CONCLUSIONS

This study provides three results of note.

- (1) Analysis of information can be abstracted from monetary values in special cases to variability of returns in general cases.
- (2) With an appropriately chosen model one can separate and calculate independently the influences on a stochastic process of having either
  - (a) more timely information, or
  - (b) better quality information, or both.
- (3) One can bring methods of statistical inference to stochastic analysis.



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(Paul C. Kettler)

CENTRE OF MATHEMATICS FOR APPLICATIONS

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF OSLO

P.O. BOX 1053, BLINDERN

N-0316 OSLO

NORWAY

*E-mail address:* paulck@math.uio.no

*URL:* <http://www.math.uio.no/~paulck/>