

ON LOCAL TIMES: APPLICATION TO PRICING USING BID-ASK

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ABSTRACT. In this paper, we derive the evolution of a stock price from the dynamics of the “best bid” and “best ask”. Under the assumption that the bid and ask prices are described by semimartingales, we study the completeness and the possibility for arbitrage on such a market. Further, we discuss (insider) hedging for contingent claims with respect to the stock price process.

1. INTRODUCTION

The theory of asset pricing and its fundamental theorem were initiated in the Arrow-Debreu model, the Black-Scholes formula, and the Cox-Ross model. They have now been formalized in a general framework by Harrison and Kreps (9), Harrison and Pliska (10), and Kreps (13), according to the no arbitrage principle. In the classical setting, the market is assumed to be frictionless — a no arbitrage dynamic price process is a martingale under an probability measure equivalent to the reference probability measure.

However, real financial markets are not frictionless, and so an important literature on pricing under transaction costs and liquidity risk has appeared. See (1; 12) and references therein. In these papers the bid-ask spreads are explained by transaction costs. Jouini and Kallal in (12) propose an axiomatic approach in continuous time, assigning to financial assets a dynamic ask price process (respectively, a dynamic bid price process.) They proved that the absence of arbitrage opportunities is equivalent to the existence of a frictionless arbitrage-free process lying between the bid and the ask processes, *i.e.*, a process which could be transformed into a martingale under a well-chosen probability measure. The bid-ask spread in this setting can be interpreted as a transaction cost or as the result of entering buy and sell orders.

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Taking into account both transaction costs and liquidity risk Bion-Nadal in (1) changed the assumption of sublinearity of ask price (respectively, superlinearity of bid price) made in (12) to that of convexity (respectively, concavity) of the ask (respectively, bid) price. This assumption combined with the time-consistency property for dynamic prices allowed her to generalize the result of Jouini and Kallal. She proved that the “no free lunch” condition for a time-consistent dynamic pricing procedure [TCPP] is equivalent to the existence of an equivalent probability measure \mathbb{Q} that transforms a process between the bid and ask processes of any financial instrument into a martingale.

In recent years, a pricing theory has also appeared taking inspiration from the theory of risk measures. First to investigate in a static setting were Carr, Geman, and Madan (2) and Föllmer and Schied (8). The point of view of pricing via risk measures was also considered in a dynamic way using backward stochastic differential equations [BSDE] by El Karoui and Quenez (6), El Karoui, Peng, and Quenez (5), and Peng (15; 16). This theory soon became a useful tool for formulating many problems in mathematical finance, in particular for the study of pricing and hedging contingent claims (5). Moreover, the BSDE point of view gave a simple formulation of more general recursive utilities and their properties, as initiated by Duffie and Epstein (1992) in their [stochastic differential] formulation of recursive utility (5).

In the past in real financial markets the load of providing liquidity was given to market makers, specialists, and brokers, who trade only when they expect to make profits. Such profits are the price that investors and other traders pay in order to execute their orders when they want to trade. To ensure steady trading the market makers sell to buyers and buy from sellers, and get compensated by the so-called bid-ask spread. The most common price for referencing stocks is the last trade price, but the last price is not necessarily the price that a person can subsequently trade. At any given moment in a sufficiently liquid market there is a best or highest “bid” price from someone who wants to buy the stock and there is a best or lowest “ask” price from someone who wants to sell the stock. The best bid price $R(t)$ and best ask (or best offer) price $T(t)$ are the highest buying price and the lowest selling price at any time t of trading.

In the present work we consider models of financial markets in which all parties involved (buyers and sellers) find incentives to participate. Our framework is different from the existing approach. See (1; 12) and references therein, where the authors assume some properties (sublinearity, convexity, *etc.*) on the ask (respectively, bid) price function in order to define a dynamic ask (respectively, bid.) Rather, we assume that the different bid and ask prices are given. Then the question we address is how to model the best bid (respectively, the best ask) price process with the intention to obtain the stock price dynamics.

The assumption that the bid and ask processes are described by [continuous] semimartingales entails that the stock price admits arbitrage opportunities. Further, it turns out that the price process possesses the Markov property if the bid and ask are Brownian motions or processes of Ornstein-Uhlenbeck type, or more generally are Feller processes. Note that our results are obtained without assuming arbitrage opportunities.

This paper is related to (11) where the authors explore market situations where a large trader causes the existence of arbitrage opportunities for small traders in complete markets. The arbitrage opportunities considered are “hidden,” meaning that almost all are not observable to the small traders, or to scientists studying markets, because they occur on time sets of Lebesgue measure zero.

The paper is organized as follows: Section 2 presents the model. Section 3 studies the Markovian property of the processes, while Sections 4 and 5 are devoted to the study of completeness, arbitrage and insider hedging on a market driven by such processes.

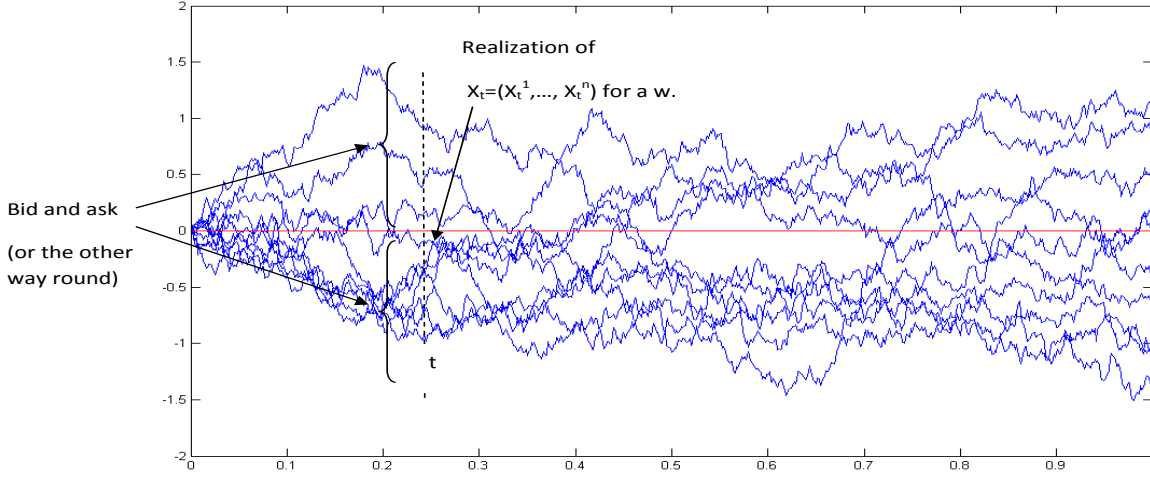
2. THE MODEL

Let $B_s = (B_s^1, \dots, B_s^n)^T$ (where $(\cdot)^T$ denotes transposition) be an n -dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

Suppose a bid and ask price process $X_i(t) \in \mathbb{R}$, $1 \leq i \leq n(t)$, is given by a stochastic diffusion differential equation of the form:

$$(2.1) \quad X_i(t) = X(0) + \int_0^t a_i(s, X_s, \omega) ds + \int_0^t \sigma_i(s, X_s, \omega) dB_i(s)$$

Here we consider the following model for bid and ask prices.



The evolution of the stock price process $S(t)$ is based on $X_i(t)$, $i = 1, \dots, n$. Denote by $\text{Bid}(t)$ the best bid and $\text{Ask}(t)$ the best ask at time t . Then $\text{Bid}(t)$ is the lowest price that a day trader seller is willing to accept for a stock at that time and $\text{Ask}(t)$ is the highest price that a day trader buyer is willing to pay for that stock at any particular point in time. Let us define the processes

$$R_t := \min\{X_i(t) : X_i(t) \geq 0, i = 1, \dots, n\}$$

$$= \min_{1 \leq i \leq n} \left\{ + \frac{\text{sgn}(X_i(t)) + 1}{2} X_i(t) \right\}$$

$$T_t := \max\{X_i(t) : X_i(t) \leq 0, i = 1, \dots, n\}$$

$$= \max_{1 \leq i \leq n} \left\{ - \frac{\text{sgn}(X_i(t)) - 1}{2} X_i(t) \right\},$$

where we use the convention that $\min\{\emptyset\} = 0$ and $\max\{\emptyset\} = 0$. Then $\text{Bid}(t)$ and $\text{Ask}(t)$ can be modeled as

$$(2.2) \quad \text{Bid}(t) := \min\{R(t), -T(t)\}$$

and

$$(2.3) \quad \text{Ask}(t) := \max\{R(t), -T(t)\}$$

Given $\text{Bid}(t)$ and $\text{Ask}(t)$, the market makers agree on a stock price within the bid/ask spread. That is

$$(2.4) \quad S(t) = \alpha(t) \text{Bid}(t) + (1 - \alpha(t)) \text{Ask}(t),$$

where $\alpha(t)$ is a stochastic process such that $0 \leq \alpha(t) \leq 1$. One could choose, *e.g.*,

$$\alpha(t) = \sigma(t)$$

for a function $\sigma: [0, T] \rightarrow [0, 1]$, or

$$\alpha(t) = f(R(t), T(t))$$

for a function $f: \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$.

For convenience, we assume that $\alpha(t) \equiv 1/2$, that is

$$(2.5) \quad \begin{aligned} S_t &= \frac{\text{Bid}(t) + \text{Ask}(t)}{2} \\ &= \frac{R(t) - T(t)}{2} \end{aligned}$$

3. MARKOVIAN PROPERTY OF PROCESSES R , T , AND S

For convenience, let us discuss briefly the Markovian property of the processes $R(t)$, $T(t)$, and $S(t)$ in some particular cases. The two cases considered here are the cases when the $\{X_t^i\}$ are Brownian motions or Ornstein-Uhlenbeck processes. Let us first have the definition of semimartingale rank processes.

Definition 3.1. Let X_1, \dots, X_n be continuous semimartingales. For $1 \leq k \leq n$, the k -th *rank process* of X_1, \dots, X_n is defined by

$$X^{(k)} = \max_{i_1 < \dots < i_k} \min(X_{i_1}, \dots, X_{i_k}),$$

where $1 \leq i_1$ and $i_k \leq n$.

Note that according to Definition 3.1 for $t \in \mathbb{R}^+$

$$\max_{1 \leq i \leq n} X_i(t) = X^{(1)}(t) \geq X^{(2)}(t) \geq \dots \geq X^{(n)}(t) = \min_{1 \leq i \leq n} X_i(t),$$

so that at any given time, the values of the rank processes represent the values of the original processes arranged in descending order, *i.e.*, the reverse order statistics.

Let $X_t^+ = \max(X_t, 0)$ and $X_t^* = \min(X_t, 0)$. Further set

$$R_t := \min_{1 \leq i \leq n} X_i^+(t)$$

$$T_t := \max_{1 \leq i \leq n} X_i^*(t)$$

Then using Definition 3.1 we get

$$(3.1) \quad \begin{aligned} R_t &:= X^{(n)+}(t) \\ T_t &:= X^{(1)*}(t) \end{aligned}$$

3.1. The Brownian motion case. We assume here that the processes $\{X_t^i\}$, $1 \leq i \leq n$, are independent Brownian motions.

Proposition 3.2. *The process R possesses the Markov property with respect to the filtration $\mathcal{F}_t := \mathcal{F}_t^B \cap \sigma(R(t); 0 \leq t \leq T)$.*

Let us employ the following useful Lemma (4).

Lemma 3.3. *Let $E = (-\infty, \infty)$, and let $\mathcal{B} = \mathcal{B}_{(-\infty, \infty)}$. Let X be a Markov process on the space (E, \mathcal{B}) with transition function $P(t, x, \Gamma)$. Assume that*

$$(3.2) \quad P(t, x, \Gamma \cap (-\Gamma)) = P(t, -x, \Gamma \cap (-\Gamma))$$

Then the process $\tilde{X} = (|X|)$ yields a Markov process on the state space $(\tilde{E}, \tilde{\mathcal{B}})$ with transition function

$$\tilde{P}(t, x, \Gamma) = P(t, x, \Gamma) + P(t, x, -\Gamma), \quad t \geq 0, \quad x \in E, \quad \Gamma \in \tilde{\mathcal{B}},$$

where $\tilde{E} = f(E)$ and $f(\mathcal{B}) \subseteq \tilde{\mathcal{B}}$, with

$$(3.3) \quad \begin{aligned} f: \mathbb{R} &\longrightarrow \mathbb{R}^+ \\ x &\longrightarrow |x| \end{aligned}$$

Proof. (of Lemma 3.3) See (4, Theorem 10.13). \square

Remark 3.4. It is clear that the transition of a n -dimensional Brownian motion satisfied Equation (3.2). Therefore, the transformation of Equation (3.3) can be applied to the Brownian motion.

Proof. (of Proposition 3.2) We first prove that $B^+ = \max(B, 0)$ is a Markov process. We have $B^+ = \frac{1}{2}(|B| + B)$. The second term in the right hand side is Markovian, and by Lemma 3.3 the first term is also Markovian. We conclude that B^+ is Markovian as the sum of two Markovian processes. See, *e.g.*, (4, p. 327, Remark 1).

Now, we proceed by induction. The case $n = 1$ is trivial; start then with $n = 2$. $R_t = \min(B_1^+(t), B_2^+(t))$, and we can rewrite $R_t = B_1^+(t) - (B_1^+(t) - B_2^+(t))^+$, but

$$(B_1^+(t) - B_2^+(t))^+ = \frac{1}{2} [|B_1^+(t) - B_2^+(t)| + (B_1^+(t) - B_2^+(t))]$$

by Lemma 3.3. We conclude that R is a Markov process. Now assume that the result holds for some n . Given Markov processes $B_1^+, \dots, B_n^+, B_{n+1}^+$, we define $B^{(k)}$, $k = 1, \dots, n$, as above, and also set

$$B^{[k]} = \max_{1 \leq i_1 < \dots < i_k \leq n+1} \min(B_{i_1}, \dots, B_{i_k})$$

The process $B^{[k]}(\cdot)$ is the k -th-rank process with respect to all $n + 1$ Markov processes B_1, \dots, B_{n+1} . It is also convenient to set $B_{(0)}(\cdot) = +\infty$. We can write $B^{[1]+} = \min(B^{(1)+}, B_{n+1})$, which leads us to the case $n = 2$. The desired result follows by induction. \square

Proposition 3.5. *The process T possesses the Markov property with respect to the filtration $\mathcal{F}_t := \mathcal{F}_t^B \cap \sigma(R(t); 0 \leq t \leq T)$.*

Proof. See the proof of Proposition 3.2. \square

Corollary 3.6. *The process S possesses the Markov property with respect to the filtration $\mathcal{F}_t := \mathcal{F}_t^B \cap \sigma(R(t); 0 \leq t \leq T)$.*

Proof. The process Z defined by $Z_t = R_t + T_t$ for all $t \geq 0$ is a Markov process as the sum of two Markov processes. \square

3.2. The Ornstein-Uhlenbeck case. Here we assume that the process $X(t) = (X_1(t), \dots, S_n(t))$ is an n -dimensional Ornstein-Uhlenbeck process. That is

$$dX_i(t) = -\alpha_i X_i(t) dt + \sigma_i dB_i(t), \quad 1 \leq i \leq n,$$

where α_i and σ_i are parameters.

Proposition 3.7. *The processes R , T , and S defined by Equations (2.5) and (3.1) possess the Markov property.*

Proof. The conclusion follows from the proof of Proposition 3.2 using Lemma 3.3. \square

Remark 3.8. Using continuous and open transformations of Markov processes, the above results can be generalized to the case when the bid and ask processes are Feller processes. See (4).

4. FURTHER PROPERTIES OF S_t

In this Section, we want to use the semimartingale decomposition of our price process S_t to analyze completeness and arbitrage on a market driven by such a process. We need the following result. See (7, Proposition 4.1.11).

Proposition 4.1. *Let X_1, \dots, X_n be continuous semimartingales of the form Equation (2.1). For $k \in \{1, 2, \dots, n\}$, let $u(k) = (u_t(k), t \geq 0) : \Omega \times [0, \infty) \rightarrow \{1, 2, \dots, n\}$ be any predictable process with the property*

$$X^{(k)}(t) = X_{u_t(k)}(t)$$

Then the k -th rank processes $X^{(k)}$, $k = 1, \dots, n$, are semimartingales, and we have

$$\begin{aligned}
(4.1) \quad X^{(k)}(t) &= X^{(k)}(0) + \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} dX_i(s) \\
&\quad + \frac{1}{2} \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} d_s L_s^0((X^{(k)} - X_i)^+) \\
&\quad - \frac{1}{2} \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} d_s L_s^0((X^{(k)} - X_i)^-),
\end{aligned}$$

where $L_t^0(X)$ is the local time of the semimartingale X at zero, defined by

$$|X_t| = |X_0| + \int_0^t \operatorname{sgn}(X_{s-}) dX_s + L_t^0(X)$$

Proof. We find that

$$(4.2) \quad X_t^{(k)} - X_0^{(k)} = \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} dX_s^i + \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} d(X_s^{(k)} - X_s^i),$$

where we used the property $\sum_{i=1}^n \mathbf{1}_{\{u_s(k)=i\}} = 1$. It follows that

$$\begin{aligned}
X_t^{(k)} - X_0^{(k)} &= \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} dX_s^i \\
&\quad + \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} d(X_s^{(k)} - X_s^i)^+ \\
&\quad - \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} d(X_s^{(k)} - X_s^i)^-
\end{aligned}$$

We note the fact

$$(4.3) \quad \{u_s(k) = i\} \subset \{X_s^{(k)} = X_i(s)\}$$

We now use the following formula:

$$(4.4) \quad \frac{1}{2}L_t^0(X) = \int_0^t \mathbf{1}_{\{X_s=0\}} dX_s,$$

which is valid for non-negative semimartingales X .

Then, by applying Equation (4.4) to $(X^{(k)}(t) - X_i(t))^\pm$, $t \geq 0$, Equation (4.2) becomes:

$$\begin{aligned} X^{(k)}(t) &= X^{(k)}(0) + \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} dX_i(s) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} d_s L_s^0((X^{(k)} - X_i)^+) \\ &\quad - \frac{1}{2} \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} d_s L_s^0((X^{(k)} - X_i)^-) \end{aligned}$$

□

4.1. The Brownian motion case. If $X_i(t) = B_i^+(t)$ or $B_i^*(t)$, $i = 1, \dots, n$ are n independent Brownian motions, the evolution of R_t and T_t follows from Theorem 4.1.

Corollary 4.2. *Let the processes R_t and T_t be given by Equations (3.1). Then $R_t = B^{(n)+}(t)$ and $T_t = B^{(1)*}(t)$, and we have*

$$\begin{aligned} R(t) &= R(0) + \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(n)=i\}} \left\{ dB_i^+(s) - \frac{1}{2} d_s L_s^0(B_i^+ - R) \right\} \\ &= R(0) + \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(n)=i\}} \left\{ \mathbf{1}_{\{B_i(s)>0\}} dB_i(s) + \frac{1}{2} [d_s L_s^0(B_i) - d_s L_s^0(B_i^+ - R)] \right\} \end{aligned}$$

and

$$\begin{aligned} T(t) &= T(0) + \sum_{i=1}^n \int_0^t \mathbf{1}_{\{v_s(n)=i\}} \left\{ dB_i^*(s) + \frac{1}{2} d_s L_s^0(T - B_i^*) \right\} \\ &= T(0) + \sum_{i=1}^n \int_0^t \mathbf{1}_{\{v_s(n)=i\}} \left\{ \mathbf{1}_{\{B_i(s) \leq 0\}} dB_i(s) + \frac{1}{2} [d_s L_s^0(T - B_i^*) - d_s L_s^0(B_i)] \right\} \end{aligned}$$

We can rewrite R_t and T_t as follows:

$$R_t = R_0 + M_t^R + V_t^R$$

$$T_t = T_0 + M_t^T + V_t^T$$

where $\{M_t^R, M_t^T\}$ are continuous local martingales and $\{V_t^R, V_t^T\}$ are continuous processes of locally bounded variation given by

$$(4.5) \quad V_t^R = \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(n)=i\}} \frac{1}{2} [d_s L_s^0(B_i) - d_s L_s^0(B_i^+ - R)],$$

$$(4.6) \quad M_t^R = \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(n)=i\}} \mathbf{1}_{\{B_i(s) > 0\}} dB_i(s),$$

$$(4.7) \quad V_t^T = \sum_{i=1}^n \int_0^t \mathbf{1}_{\{v_s(n)=i\}} \frac{1}{2} [d_s L_s^0(T - B_i^*) - d_s L_s^0(B_i)],$$

$$(4.8) \quad M_t^T = \sum_{i=1}^n \int_0^t \mathbf{1}_{\{v_s(n)=i\}} \mathbf{1}_{\{B_i(s) \leq 0\}} dB_i(s).$$

The following Corollary gives the semimartingale decomposition satisfied by the process S_t .

Corollary 4.3. *Assume that the process S_t is given by Equation (2.5). Then one can write $S_t = f(A_t)$, where $A_t = (R_t, T_t)$ and $f(x_1, x_2) = \frac{1}{2}(x_1 - x_2)$, and we have:*

$$\begin{aligned}
(4.9) \quad S(t) &= S(0) + \frac{1}{2} \sum_{i=1}^n \int_0^t (\mathbf{1}_{\{u_s(n)=i\}} \mathbf{1}_{\{B_i(s)>0\}} - \mathbf{1}_{\{v_s(n)=i\}} \mathbf{1}_{\{B_i(s)\leq 0\}}) dB_i(s) \\
&\quad + \frac{1}{2} \sum_{i=1}^n \int_0^t (\mathbf{1}_{\{u_s(n)=i\}} + \mathbf{1}_{\{v_s(n)=i\}}) d_s L_s^0(B_i) \\
&\quad - \frac{1}{2} \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(n)=i\}} d_s L_s^0(B_i^+ - R) + \mathbf{1}_{\{v_s(n)=i\}} d_s L_s^0(T - B_i^*)
\end{aligned}$$

In order to price options with respect to $S(t)$ one should ensure that $S(t)$ does not admit arbitrage possibilities. The natural question which arises at this point is the following: Can we find an equivalent probability measure \mathbb{Q} such that S is a \mathbb{Q} -sigma martingale (see (17) for definitions)? Since our process S is continuous we can reformulate the question as: Can we find an equivalent probability measure \mathbb{Q} such that S is a \mathbb{Q} -local martingale?¹

We first give the following useful remark which is a part of (18, Theorem 1).

Remark 4.4. Let $X = X_0 + M_t + V_t$ be a continuous semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Let $C_t = [X, X]_t = [M, M]_t$, $0 \leq t \leq T$. A necessary condition of existence for an equivalent martingale measure is that $dV \ll dC$.

Consequence 4.5. *Since local time is singular, we observe that the total variation of the bounded variation part in Equation (4.9) cannot be absolutely continuous with respect to the quadratic variation of the martingale. It follows that the set of equivalent martingale measures is empty and thus such a market contains arbitrage opportunities.*

4.2. (In)complete market with hidden arbitrage. We consider a model where $S_t|_{t>0}$ denotes a stochastic process modeling the price of a risky asset, and $R_t|_{t>0}$ denotes the value of a risk-free money market account. We assume a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t|_{t>0}, \mathbb{P})$, where $\mathcal{F}_t|_{t>0}$ satisfies the ‘‘usual hypotheses.’’ In such a market, a trading strategy (a, b) is

¹In fact, since S is continuous and since all sigma-martingales are local martingales we only need to concern ourselves with local martingales.

self-financing if a is predictable, b is optional, and

$$(4.10) \quad a(t)S(t) + b(t)R(t) = a(0)S(0) + b(0)R(0) + \int_0^t a(s) dS(s) + \int_0^t b(s) dR(s)$$

for all $0 \leq t \leq T$. We take $S_0 = 0$ and $R(t) \equiv 1$ (thus the interest rate $r = 0$) so that $dR(t) = 0$, and Equation (4.10) becomes

$$a(t)S(t) + b(t)R(t) = b(0) + \int_0^t a(s) dS(s)$$

Definition 4.6 (See (11)).

- (1) We call a random variable $H \in \mathcal{F}_T$ a *contingent claim*. Further, a contingent claim H is said to be \mathbb{Q} -*redundant* if for a probability measure \mathbb{Q} there exists a self-financing strategy (a, b) such that

$$V_t^{\mathbb{Q}} = \mathbb{E}_{\mathbb{Q}}[H | \mathcal{F}_t] = b_0 + \int_0^t a_s dS_s,$$

where $V(t)|_{t>0}$ is the value of the portfolio.

- (2) A market $(S(t), R(t)) = (S(t), 1)$ is \mathbb{Q} -*complete* if every $H \in L^1(\mathcal{F}_T, d\mathbb{Q})$ is \mathbb{Q} -redundant.

Define the process $M^S(t)|_{t>0}$ as follows.

$$(4.11) \quad M^S(t) = \frac{1}{2} \sum_{i=1}^n \int_0^t (\mathbf{1}_{\{u_s(n)=i\}} \mathbf{1}_{\{B_i(s)>0\}} - \mathbf{1}_{\{v_s(n)=i\}} \mathbf{1}_{\{B_i(s)\leq 0\}}) dB_i(s)$$

Then the following Theorem is immediate from (11, Theorem 3.2).

Proposition 4.7. *Suppose there exists a unique \mathbb{P}^* equivalent to \mathbb{P} such that $M^S(t)$ is a \mathbb{P}^* -local martingale. Then the market $(S(t), 1)$ is \mathbb{P}^* -complete.*

Proof. Omitted

Proposition 4.8. *Suppose that $n \geq 2$. Then, there exists no unique martingale measure \mathbb{P}^* such that $M^S(t)$ is a \mathbb{P}^* -local martingale.*

Proof. Because of Equation(4.11), we observe that $M^S(t)$ is a \mathbb{P} -martingale. Let us construct another equivalent martingale measure \mathbb{P}^* . For this purpose assume wlog that $u_s(n)$ and $v_s(n)$ are given by

$$u_s(n) = \min \{i \in \{1, \dots, n\} : B_i^+(t) = R(t)\}$$

and

$$v_s(n) = \min \{i \in \{1, \dots, n\} : B_i^*(t) = T(t)\}$$

Now define the process h as

$$h(t) = \mathbf{1}_{\{A(t)\}},$$

where

$$A(t) = \{\omega \in \Omega : \beta(t, \omega) = 0\},$$

with

$$(4.12) \quad \beta(s) = \sum_{i=1}^n \left(\mathbf{1}_{\{u_s(n)=i\}} \mathbf{1}_{\{B_i(s)>0\}} - \mathbf{1}_{\{v_s(n)=i\}} \mathbf{1}_{\{B_i(s)\leq 0\}} \right)$$

One finds that $\Pr[A(t)] > 0$ for all t . Let us define the equivalent measure \mathbb{P}^* with respect to a density process Z_t given by

$$Z_t = \mathbb{E}[N_t]$$

Here $\mathbb{E}(N_t)$ denotes the Doléans-Dade exponential of the martingale N_t defined by

$$N_t = \sum_{i=1}^n \int_0^t h(s) dB_i(s)$$

Then it follows from the Girsanov-Meyer theorem (see (17)) that $M^S(t)$ has a \mathbb{P}^* -semimartingale decomposition with a bounded variation part given by

$$\int_0^t 2h(s) d\langle M^S, M^S \rangle_s$$

We have that

$$\int_0^t 2h(s) d\langle M^S, M^S \rangle_s = \frac{1}{2} \int_0^t h(s)\beta(s) ds$$

Since $h\beta = 0$ it follows that

$$\int_0^t h(s) d\langle M^S, M^S \rangle_s = 0$$

Thus $M^S(t)$ is a \mathbb{P}^* -martingale. Since \mathbb{P} is also a martingale measure with $\mathbb{P} \neq \mathbb{P}^*$ the proof follows. \square

Remark 4.9. In the case $n = 1$ (a single Bid/Ask) the market becomes complete since the process $\beta(t)$, defined by Equation (4.12) in the proof is equal to $\text{sgn}(B(t))$. Therefore the unique martingale measure is \mathbb{P} .

We can then deduce the following theorem on our process $S(t)$.

Proposition 4.10. *Suppose that $S = S(t)|_{t \geq 0}$ is given by Equation (4.9), and $M^S(t)$ is given by Equation (4.11). Then*

- (1) *For $n = 1$ (a single Bid/Ask) the market $(S(t), 1)$ is \mathbb{P} -complete and admits the arbitrage opportunity of Equation (4.13).*
- (2) *For $n \geq 2$ (more than a single Bid/Ask) the market $(S(t), 1)$ is incomplete and arbitrage exists.*

Proof. From Theorem 4.8, we know that the market is \mathbb{P} -complete for $n = 1$ and incomplete for $n > 1$. Let \mathbb{P} be such that $M^S(t)$ is a \mathbb{P} -local martingale.

For $n = 1$, let us construct an arbitrage strategy. Let

$$(4.13) \quad a_s = \mathbf{1}_{\{\text{supp}(d[M^S, M^S])\}^c}(s),$$

where $\text{supp}(d[M^S, M^S])$ denotes the ω by ω support of the (random) measure $d[M^S, M^S]_s(\omega)$; that is, for fixed ω it is the smallest closed set in \mathbb{R}_+ such that $d[M^S, M^S]_s$ does not charge its complement. Compare the proof of Proposition 4.8.

Let

$$H = H(T) = \frac{1}{2} \sum_{i=1}^n \int_0^T (\mathbf{1}_{\{u_s(n)=i\}} + \mathbf{1}_{\{v_s(n)=i\}}) d_s L_s^0(B_i) \\ - \frac{1}{2} \sum_{i=1}^n \left\{ \int_0^T \mathbf{1}_{\{u_s(n)=i\}} d_s L_s^0(B_i^+ - R) + \int_0^T \mathbf{1}_{\{v_s(n)=i\}} d_s L_s^0(T - B_i^*) \right\}$$

Assume wlog that $H \in L^1(\mathbb{P})$. Then by Theorem 4.7 there exists a self financing strategy (j_t, b) such that

$$H = H(T) = \mathbb{E}[H(T)] + \int_0^T j(s) dS(s).$$

However, by Equation (4.13) we also have

$$H(T) = 0 + \int_0^T a(s) dH(s).$$

Moreover, we have $\int_0^t a(s) dM^S(s) = 0$, $0 \leq t \leq T$, by construction of the process a . Hence,

$$H = H(T) = 0 + \int_0^T a(s) dS(s),$$

which is an arbitrage opportunity. □

5. PRICING AND INSIDER TRADING WITH RESPECT TO $S(t)$

In this Section we discuss a framework introduced in (3), which enables us pricing of contingent claims with respect to the price process $S(t)$ of the previous Sections. We even consider the case of insider trading, that is the case of an investor who has access to inside information. To this end we need some notions.

We consider a market driven by the stock price process $S(t)$ on a filtered probability space $(\Omega, \mathcal{H}, (\mathcal{H}_t)_{t \geq 0}, \mathbb{P})$. We assume that the decisions of the trader are based on market information given by the filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$ with $\mathcal{H}_t \subset \mathcal{G}_t$ for all $t \in [0, T]$, $T > 0$ being a fixed terminal time. In this context an insider strategy is represented by a \mathcal{G}_t -adapted process $\varphi(t)$ and we interpret all anticipating integrals as the forward integral defined in (14; 19).

In such a market, a natural tool to describe the self-financing portfolio is the forward integral of an integrand process Y with respect to an integrator S , denoted by $\int_0^t Y d^-S$. See the Appendix. The following definitions and concepts are consistent with those given in (3).

Definition 5.1. A *self-financing* portfolio is a pair (V_0, a) , where V_0 is the initial value of the portfolio and a is a \mathcal{G}_t -adapted and S -forward integrable process specifying the number of shares of S held in the portfolio. The market value process V of such a portfolio at time $t \in [0, T]$, is given by

$$(5.1) \quad V(t) = V_0 + \int_0^t a(s) d^-S(s),$$

while $b(t) = V(t) - S(t)a(t)$ constitutes the number of shares of the less risky asset held.

5.1. \mathcal{A} -martingales. Now we briefly review the definition of \mathcal{A} -martingales which generalizes the concept of a martingale. We refer to (3) for more information about this notion. Throughout this Section \mathcal{A} is a real linear space of measurable processes indexed by $[0, 1)$ with paths bounded on each compact interval of $[0, 1)$.

Definition 5.2. A process $X = (X(t), 0 \leq t \leq T)$ is said to be an \mathcal{A} -martingale if every $\theta \in \mathcal{A}$ is X -improperly forward integrable (see Appendix) and

$$(5.2) \quad \mathbb{E} \left[\int_0^t \theta(s) d^-X(s) \right] = 0 \quad \text{for every } t, 0 \leq t \leq T$$

Definition 5.3. A process $X = (X(t), 0 \leq t \leq T)$ is said to be an \mathcal{A} -semimartingale if it can be written as the sum of an \mathcal{A} -martingale M and a bounded variation process V , with $V(0) = 0$.

Remark 5.4.

- (1) Let X be a continuous \mathcal{A} -martingale with X belonging to \mathcal{A} , then, the quadratic variation of X exists improperly. In fact, if $\int_0^\cdot X(t) d^-X(t)$ exists improperly, then one can show that $[X, X]$ exists improperly and $[X, X] = X^2 - X^2(0) - 2 \int_0^\cdot X(s) d^-X(s)$. See (3) for details.
- (2) Let X be a continuous square integrable martingale with respect to some filtration \mathcal{F} . Suppose that every process in \mathcal{A} is the restriction to $[0, T)$ of a process $(\theta(t), 0 \leq t \leq T)$ which is \mathcal{F} -adapted. Moreover, suppose that its paths are left continuous with right limits and $E[\int_0^T \theta^2(t) d[X]_t] < \infty$. Then X is an \mathcal{A} -martingale.

5.2. Completeness and arbitrage: \mathcal{A} -martingale measures. We first recall some definitions and notions introduced in (3).

Definition 5.5. Let h be a self-financing portfolio in \mathcal{A} which is S -improperly forward integrable and X its wealth process. Then h is an \mathcal{A} -arbitrage if $X(T) = \lim_{t \rightarrow T} X(t)$ exists almost surely, $\Pr[X(T) \geq 0] = 1$ and $\Pr[X(T) > 0] > 0$.

Definition 5.6. If there is no \mathcal{A} -arbitrage, the market is said to be \mathcal{A} -arbitrage free.

Definition 5.7. A probability measure $Q \sim P$ is called a \mathcal{A} -martingale measure if with respect to Q the process S is an \mathcal{A} -martingale according to Definition 5.2.

We need the following assumption. See (3).

Assumption 5.8. *Suppose that for all h in \mathcal{A} the following condition holds: h is S -improperly forward integrable and*

$$(5.3) \quad \int_0^\cdot d^- \int_0^t h(s) d^-S(s) = \int_0^\cdot h(t) d^-S(t) = \int_0^\cdot h(t) d^- \int_0^t d^-S(s)$$

The proof of the following Proposition can be found in (3).

Proposition 5.9. *Under Assumption 5.8 if there exists an \mathcal{A} -martingale measure Q , the market is \mathcal{A} -arbitrage free.*

Definition 5.10. A *contingent claim* is an \mathcal{F} -measurable random variable. Let \mathcal{L} be the set of all contingent claims of interest to the investor.

Definition 5.11.

- (1) A contingent claim C is called \mathcal{A} -*attainable* if there exists a self-financing trading portfolio $(X(0), h)$, with $h \in \mathcal{A}$, which is S -improperly forward integrable, and whose terminal portfolio value coincides with C , *i.e.*,

$$\lim_{t \rightarrow T} X(t) = C, \quad \mathbb{P}\text{-a.s.}$$

Such a portfolio strategy h is called a *replicating* or *hedging portfolio* for C , and $X(0)$ is the *replication price* for C .

- (2) A \mathcal{A} -arbitrage free market is called $(\mathcal{A}, \mathcal{L})$ -*complete* if every contingent claim in L is attainable.

Assumption 5.12. For every \mathcal{G}_0 -measurable random variable η , and h in \mathcal{A} the process $u = h(\eta)$ belongs to \mathcal{A} .

Proposition 5.13. Suppose that the market is \mathcal{A} -arbitrage free, and that Assumption 5.8 is realized. Then the replication price of an attainable contingent claim is unique.

Proof. Let \mathbb{Q} be a given measure equivalent to \mathbb{P} . For such a \mathbb{Q} , let \mathcal{A} be a set of all strategies (\mathcal{G}_t -adapted) such that Equation (5.2) in definition 5.2 is satisfied. Then, it follows from Proposition 5.9 that our market $(S(t), 1)$ in Section 4.2 is \mathcal{A} -arbitrage free. \square

In the final Section, we shall discuss attainability of claims in connection with a concrete set \mathcal{A} of trading strategies.

5.3. Hedging with respect to $S(t)$. In this Section, we want to determine hedging strategies for a certain class of European options with respect to the price process $S(t)$ of Section 4.2. Let us now assume that $n = 1$ (a single Bid/Ask). Then, the price process S is the sum of a Wiener process and a continuous process with zero quadratic variation; moreover, we have that $d[S]_t = (1/4)\beta^2(t) = 1/4$, where $\beta(t)$ is given by Equation (4.12). We can derive the following proposition which is similar to (3, Proposition 5.29).

Proposition 5.14. *Let ψ be a function in $C^0(\mathbb{R})$ of polynomial growth. Suppose that there exists $v(t, x)$, $0 \leq t \leq T$, $x \in \mathbb{R}$, of class $C^{1,2}([0, T] \times \mathbb{R}) \cap C^0([0, T] \times \mathbb{R})$ which is a solution of the following Cauchy problem.*

$$\begin{cases} \partial_t v(t, x) + \frac{1}{8} \partial_{yy} v(t, y) = 0 & \text{on } [0, T] \times \mathbb{R} \\ v(T, y) = \psi(y) \end{cases}$$

Set

$$h(t) = \partial_y(t, S(t)), \quad 0 \leq t \leq T, \quad X(0) = v(0, S(0))$$

Then $(X(0); h)$ is a self-financing portfolio replicating the contingent claim $(\psi S(T))$.

In particular, $(S(t), 1)$ is \mathcal{A}, \mathcal{L} -complete, where \mathcal{A} is given by

$$\mathcal{A} = \{ \phi(t, S(t)), \quad 0 \leq t \leq T \mid \phi: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$$

is Borel measurable, of polynomial growth, and lower bounded $\}$,

and \mathcal{L} by all claims as stated in this Proposition.

Proof. The proof is a direct consequence of Itô's Lemma for forward integrals. See (3, Proposition 5.29). □

APPENDIX

Forward integration. For the convenience of the reader, we recall some basic definitions and fundamental results about forward integration theory which have been introduced in (19; 20). In what follows $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space. Let $X = (X_t, t \in [0, T])$, $Y = (Y_t, t \in [0, T])$ be two continuous processes. We assume that all filtrations fulfill the usual conditions.

Definition 5.15. Let $X = (X(t), t \in [0, T])$ denote a continuous stochastic process and $Y = (Y(t), t \in [0, T])$ a process with path in $L^\infty([0, T])$. The ϵ -forward integral (respectively ϵ -backward, ϵ -symmetric integrals and the ϵ -covariation) is defined as follows:

$$\begin{aligned} I^-(\epsilon, Y, dX)(t) &:= \int_0^t Y(s) \frac{X(s+\epsilon) - X(s)}{\epsilon} ds, \\ I^+(\epsilon, Y, dX)(t) &:= \int_0^t Y(s) \frac{X(s) - X(s-\epsilon)}{\epsilon} ds, \\ I^0(\epsilon, Y, dX)(t) &:= \int_0^t Y(s) \frac{X(s+\epsilon) - X(s-\epsilon)}{2\epsilon} ds, \\ C_\epsilon(X, Y)(t) &:= \frac{1}{\epsilon} \int_0^t (X(s+\epsilon) - X(s))(Y(s+\epsilon) - Y(s)) ds \end{aligned}$$

Observe that these four processes are continuous.

Definition 5.16.

- (1) A family of processes $\left\{ H_t^{(\epsilon)} \right\}_{t \in [0, T]}$ is said to converge to a process $(H_t)_{t \in [0, T]}$ *uniformly on compacts in probability* (abbreviated *ucp*) if $\sup_{0 \leq t \leq T} \left| H_t^{(\epsilon)} - H_t \right| \rightarrow 0$ in probability as $\epsilon \rightarrow 0$.

- (2) The *forward, backward, symmetric integrals* and the *covariation process* are defined by the following limits in the *ucp* sense whenever they exist:

$$\begin{aligned}
 \int_0^t Y(s) d^- X(s) &:= \lim_{\epsilon \downarrow 0} I^-(\epsilon, Y, dX)(t), \\
 \int_0^t Y(s) d^+ X(s) &:= \lim_{\epsilon \downarrow 0} I^+(\epsilon, Y, dX)(t), \\
 \int_0^t Y(s) d^0 X(s) &:= \lim_{\epsilon \downarrow 0} I^0(\epsilon, Y, dX)(t), \\
 (5.4) \quad [X, Y](t) &:= \lim_{\epsilon \downarrow 0} C_\epsilon(X, Y)(t),
 \end{aligned}$$

When $X = Y$ we often put $[X, X] = [X]$.

Definition 5.17.

- (1) If $[X]$ exists, then it is always increasing, and X is said to be a *finite quadratic variation process*. The value $[X]$ is called the *quadratic variation* of X .
- (2) If $[X] = 0$, then X is called a *zero quadratic variation process* (or a *zero-energy process*).
- (3) We say that an m -dimensional process $X = (X^1, \dots, X^m)$ has all the *mutual brackets* if $[X^i, X^j]$ exists for any $i, j = 1, \dots, m$.

We recall now some basic facts which are contained in (19; 20).

Remark 5.18.

- (1) If X, Y are two continuous semimartingales, then $[X, Y] = \langle X, Y \rangle$.
- (2) If $X = Y$, then $\langle X, X \rangle$ is the quadratic variation of X and it is an increasing process. In the paper we set $\langle X, X \rangle = [X]$.
- (3) If A is a zero quadratic variation process and X is a finite quadratic variation process, then $[X, A] \equiv 0$.
- (4) A bounded variation process is a zero quadratic variation process.
- (5) If V is a bounded variation process, then $[X, V] \equiv 0$. As a consequence, if X, Y are two continuous processes such that $[X, Y]$ exists, then $[[X, Y], Z] \equiv 0$ for every continuous process Z .

Definition 5.19. Let $X = (X_t, 0 \leq t \leq T)$, $Y = (Y_t, 0 \leq t \leq T)$ be processes with paths respectively in $C^0([0, T])$ and $L_{\text{loc}}^1([0, T])$, i.e., $\int_0^t |Y(s)| ds < \infty$ for all $t < T$.

- (1) If $YI_{[0, T]}$ is X -forward integrable for every $0 \leq t < T$, then Y is said to be locally X -forward integrable on $[0, T]$. In this case there exists a continuous process which coincides on every compact interval $[0, t]$ of $[0, T]$ with the forward integral of $Y_{[0, t]}$, with respect to X . That process is denoted by $I(\cdot, Y, dX) = \int_0^\cdot Y d^-X$.
- (2) If Y is locally X -forward integrable and $\lim_{t \rightarrow T} I(t, Y, dX)$ exists almost surely, then Y is said to be X -improperly forward integrable on $[0, T]$.
- (3) If the covariation process $[X, YI_{[0, t]}]$ exists for every $0 \leq t < T$, we say that the covariation process $[X, Y]$ exists locally on $[0, T]$, and is denoted by $[X, Y]$. In this case there exists a continuous process which coincides on every compact interval $[0, t]$ of $[0, T]$ with the covariation process $[X, YI_{[0, t]}]$. That process is denoted by $[X, Y]$. If $X = Y$, we say that the quadratic variation $[X, X]$ of X exists locally on $[0, T]$.
- (4) If the covariation process $[X, Y]$ exists locally on $[0, T]$ and $\lim_{t \rightarrow T} [X, Y]_t$ exists, then the limit is called the improper covariation process between X and Y and is denoted by $[X, Y]$. If $X = Y$, then we say that the quadratic variation $[X, X]$ of X exists improperly on $[0, T]$.

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