Monopoly to Competition

a stochastic process

with implications for share and option pricing in a non-competitive market

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Department of Mathematics
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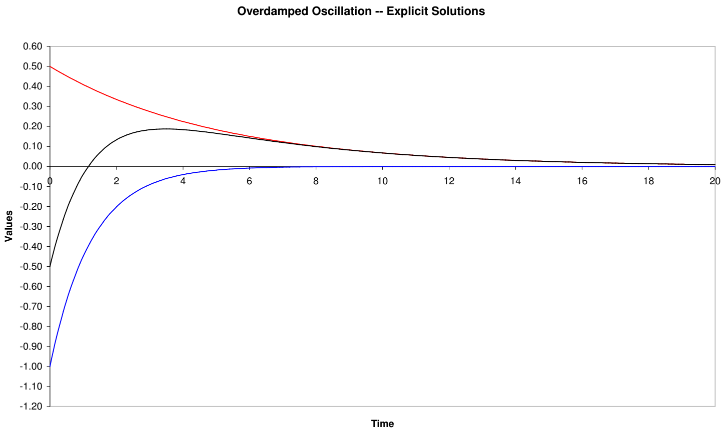
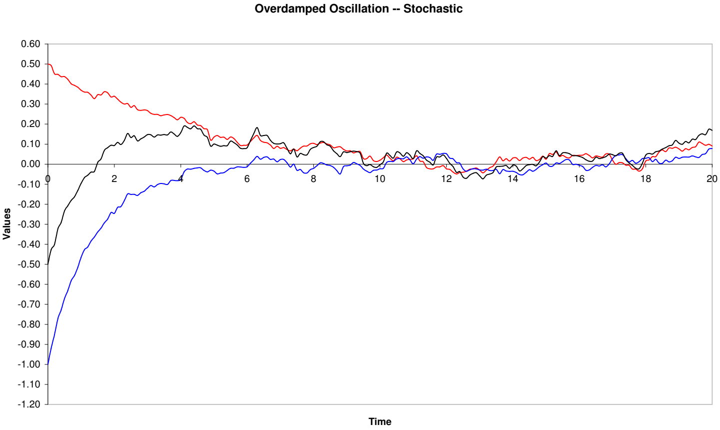
Overdamped harmonic oscillations — Deterministic



Overdamped harmonic oscillations — Stochastic #1



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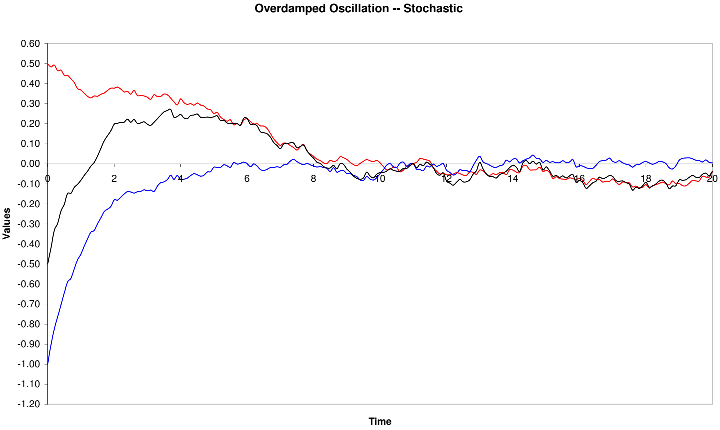
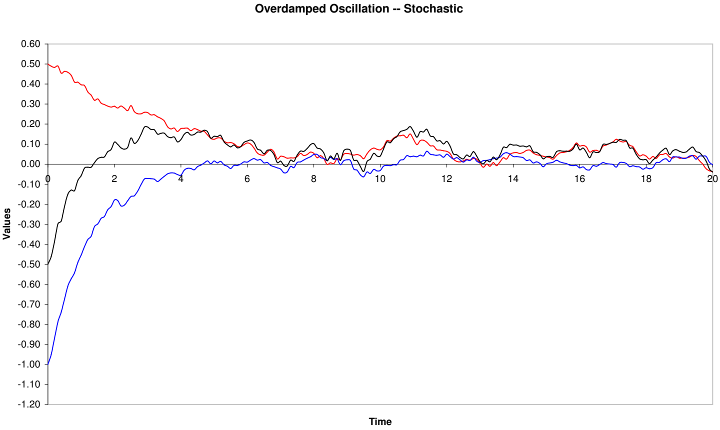
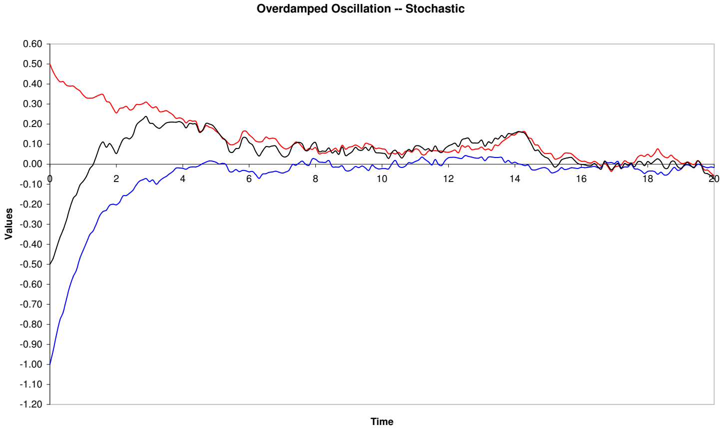
Overdamped harmonic oscillations — Stochastic #2



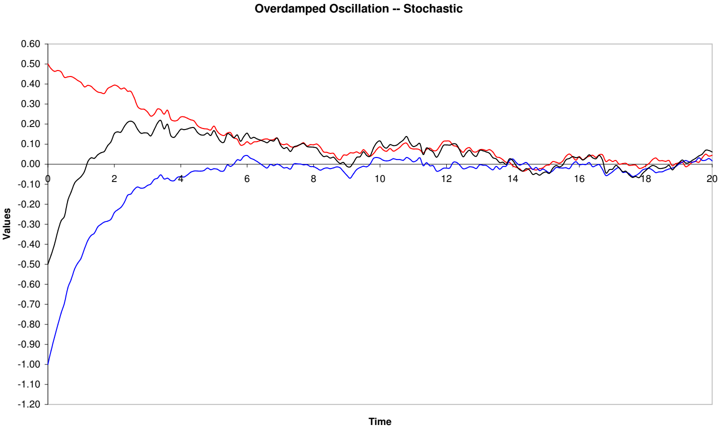
Overdamped harmonic oscillations — Stochastic #3



Overdamped harmonic oscillations — Stochastic #4



Overdamped harmonic oscillations — Stochastic #5



Background

While a rich literature exists on the states of monopoly, oligopoly, and competition, much less exists on the transitional stages, and almost nothing using stochastic analysis.

For a sample see de la Maza, M., A. Oğuş, and D. Yuret (1998, June) and Smith, G. D. (1988).

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1. limitations of the empirical record
2. challenges to formulate testable hypotheses

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- non-repeatability of experiments
  - one market response to an innovation
  - Stephen J. Gould's observation on evolution

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- difficulty of estimation for parametric models
  Response: Propose a new method.

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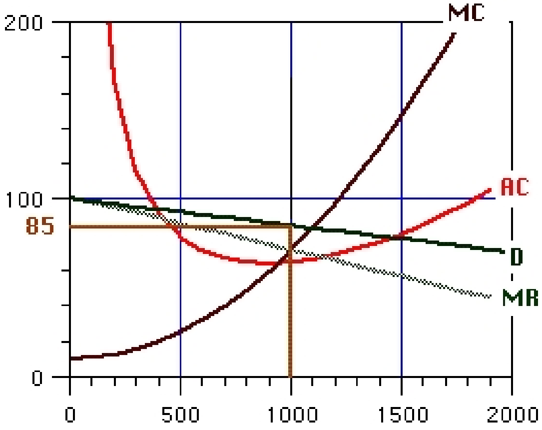
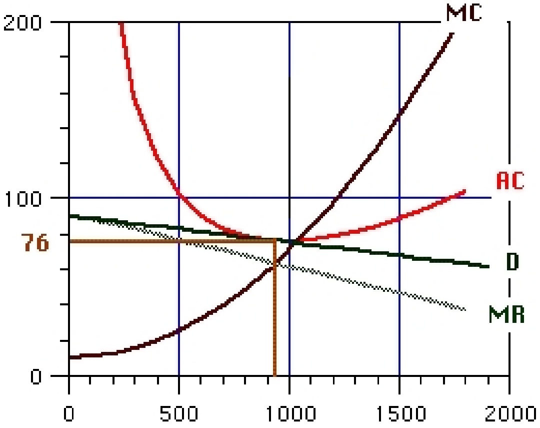
- on transitional processes from the former toward the latter

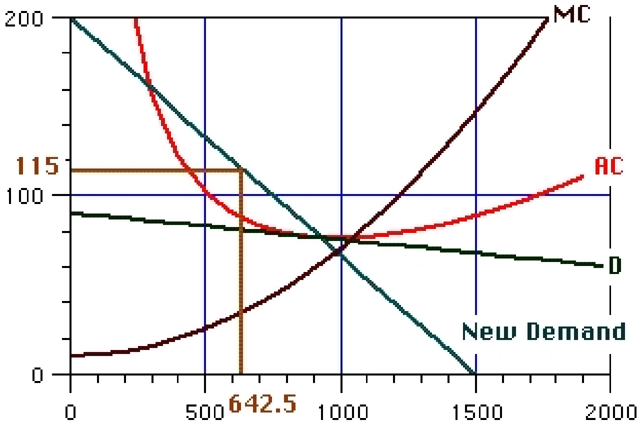
Monopolistic competition . . .



… and long-run equilibrium



A differentiated product . . .



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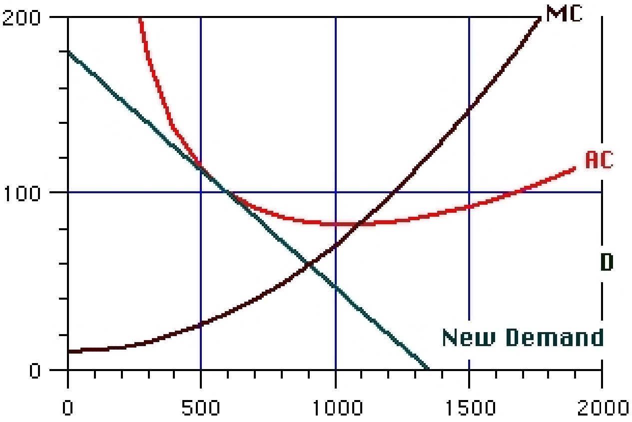

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Some monopoly calculations

The monopolist faces demand $p(q)$ with $p'(q) < 0$.

Profit equals revenue minus cost —

$$\begin{aligned}\pi(q) &= R(q) - C(q)\\ &= p(q)q - C(q)\end{aligned}$$

For maximum profit marginal revenue equals marginal cost —

$$\pi'(q) = p(q)q + p(q) - c(q) = 0$$

with $\pi''(q) < 0$.

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In particular we restrict attention to Brownian motions as sources of noise, leaving jump diffusions — including control aspects — and general Lévy processes to subsequent work.

Ornstein-Uhlenbeck

The stochastic differential equation is

$$\mathrm{d}Y(t) = -\lambda Y(t)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \quad \lambda > 0, \sigma > 0,$$

where $B(t)$ is a Brownian motion. The solution is

$$Y(t) = \mathrm{e}^{-\lambda t} Y(0) + \sigma \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}B(s)$$

Exponential decay

The stochastic differential equation is

$$\frac{\mathrm{d}Y(t)}{Y(t)} = -\lambda\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \quad \lambda > 0, \sigma > 0,$$

where $B(t)$ is a Brownian motion. The solution is

$$Y(t) = Y(0)\exp[-(\lambda + \frac{1}{2}\sigma^2)t + \sigma B(t)]$$

Damped harmonic oscillator — equation

The stochastic differential equation is

$$m\ddot{X}(t) + b\dot{X}(t) + kX(t) = aB(t)$$

Letting $\gamma = b/m$, $\omega_0^2 = k/m$, and $\alpha = a/m$, then dividing the equation above by m one has the reduced form

$$\ddot{X}(t) + \gamma\dot{X}(t) + \omega_0^2 X(t) = \alpha B(t),$$

where $B(t)$ is a Brownian motion. Recasting this equation as a coupled linear system in differential form gives

$$\mathrm{d}Y(t) = AY(t)\,\mathrm{d}t + K\,\mathrm{d}B(t), \text{ where}$$

$$Y(t) = \begin{bmatrix} X(t) \\ \dot{X}(t) \end{bmatrix}, \mathrm{d}Y(t) = \begin{bmatrix} \mathrm{d}X(t) \\ \mathrm{d}\dot{X}(t) \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{bmatrix}, K = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

Damped harmonic oscillator — solution

The solution to this equation (See Øksendal 2003, Example 5.1.3., pp. 66–67) is

$$Y(t) = \mathrm{e}^{At} Y(0) + KB(t) + AK \int_0^t \mathrm{e}^{-As} B(s)\,\mathrm{d}s$$

The system is said to be underdamped, critically damped, or overdamped, as the discriminant $\omega^2 = \omega_0^2 - \gamma^2/4$ is positive, zero, or negative, respectively.

In the underdamped case the resulting frequency of oscillation is ω, real, positive. In the overdamped case the equation has two solutions corresponding to the imaginary conjugate values of ω. The general solution is the superposition of these two.

Damped harmonic oscillator — as model

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It is this last of these now of interest.

Damped harmonic oscillator — units by context

Units for these equations, by context, are as follows.

Context	Terms — Force	m	b	k	a
Physical	dyn = g cm/s^2	g	g/s = P cm	g/s^2 =	dyn/cm
Electronic	V = H C/s^2	H	H/s = Ω	H/s^2 =	1/farad
Financial	fort = \$ y^{-1}/y^2	\$	\$/y = Φ	\$/y^2 =	fort y

In the financial context, a competitive force of one *fort* (fortis, L.) accelerates a capitalization of one dollar by one inverse year, per year per year. The unit of financial resistance is the *fohm*, for *financial ohm*, equal to one dollar per year.

Financial Momentum and Energy

One now makes an easy transition to concepts of financial momentum and energy. Momentum, as mass times velocity, is measured in *fort-years*, whereas energy, as one-half mass times velocity squared, is measured in *forts/year* or *fergs*, for *financial ergs*.

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Conservation?

Such usage now prompts the question, "Are there conservation laws for financial momentum and energy, and if so what are their implications?"

Using the equations and solutions

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We proceed apace to consider these points.

Estimating coefficients — by least squares . . .

One may estimate coefficients by a variety of methods. The principal goal is to have well-behaved residuals.

After setting initial conditions from the first three points one may perform a trivariate fit to the others by weighted least squares, estimating γ and ω_0^2, while assuming $\alpha = 0$. It is important to fit well the points more remote from zero, and also those closer to the start of the sequence. This suggests a weighting factor such as

$$w(t) = x^2(t)\mathrm{e}^{-\delta t},$$

where $x(t)$ is a point of the series, and δ is sufficiently large to render points later than a selected time insignificant. Coefficient α, then, becomes the standard deviation of the residuals.

… or by maximum likelihood

Alternatively, one may estimate the coefficients by maximum likelihood. As well, a weighting factor is appropriate, such as the one suggested above for the least-squares scheme.

General solution, overdamped harmonic oscillator

...

The case of most interest for modeling monopoly returns is that of the overdamped harmonic oscillator. This case is the linear combination of two distinct stochastic exponential decay solutions. Specifically, this is the general solution.

$$Y(t) = Y_1(t) + Y_2(t), \quad \text{where}$$
$$Y_1(t) = A_1 \exp\left[-\left(\frac{\gamma}{2} - \beta\right)\right] + \alpha B_1(t)$$
$$Y_2(t) = A_2 \exp\left[-\left(\frac{\gamma}{2} + \beta\right)\right] + \alpha B_2(t)$$

$B_1(t)$ and $B_2(t)$ are herein assumed independent Brownian motions with the same coefficient α, and $\beta = \sqrt{-\omega^2}$, real, positive.

… and simulating its paths

Simulating the paths is straightforward, as the general solution of the deterministic equation is available. One simply takes a small step, followed by a draw from the normal distribution for the Brownian increment. With initial conditions reset, the procedure iterates.

Numerical methods

No need appears for numerical methods at this stage of research. As a test, a second-order Euler integration was performed, with results accurate to about 3.5 decimal digits through the full term of evaluation.

Further model building, however, requiring numerical methods, may ensue. Euler integration will most likely suffice, for any inaccuracies it may exhibit are minor compared to the influence of the stochastic terms.

Nonetheless, more sophisticated methods such as the Runge-Kutta fourth order (RK4) or the Runge-Kutta-Fehlberg fourth/fifth order (RKF45), remain available. Stiffness is not an issue.

The monopoly bond

$$\mathrm{d}S_M = Y(t)S_M(t)\,\mathrm{d}t$$

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2. Is there a martingale measure?
3. What are option prices?

A comparison to insider trading

A monopolist and an insider have some things in common — most importantly that each is in position to earn extraordinary profits.

The main difference is this: An insider has extra information, providing the advantage; the monopolist has more influence, providing the advantage. With an insider market one speaks of nested filtrations; in a monopoly market all have the same filtration (it is assumed,) only disparate influence.

Final thoughts

Economics is a behavioral discipline. Some would (and do) say it is a science. However one always must remain aware that physical models can never provide certainty to human behavior. Many cannot provide certainty in the physical world, *e.g.*, consider quantum interactions.

However, stochastic analysis is the formalizing of uncertainty, and so stands ready to make useful statements about economic phenomena, difficulties notwithstanding.

For additional reading…

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