Potential Contributions of Copula Theory in Risk Management and Finance

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19th Century risk management

Prologue

"Until recently, credit risk management has changed little since the days of Rothschild. He always lent to both sides in any war. The winner would always ensure that both debts were repaid. If the borrower didn't repay, then Rothschild might finance a new war against the winner to enforce his debt. The cunning man was well versed in risk management."

— Satayjit Das, "Credit crash?"
Wilmott blog, February 2007

Dependency relationships of random variables

- Concordance measures …
  - are statistics —
  - Examples: Pearson's correlation, Kendall's tau, Spearman's rho, Blomquist's beta, and the tail index.
- Copula information …
  - is complete —
  - Examples: Probability copula, Lévy copula. With marginal distributions they are invertible.

Probability measure and Lévy measure, compared

distribution functions *vs.* tail integrals

$$A_x = (-\infty, x] \times \mathbb{R},\ x \in \mathbb{R} \qquad C_x = (x, \infty] \times \overline{\mathbb{R}}_+,\ x > 0$$

$$B_y = \mathbb{R} \times (-\infty, y],\ y \in \mathbb{R} \qquad D_y = \overline{\mathbb{R}}_+ \times (y, \infty],\ y > 0$$

$$F(x,y) = \int_{A_x \cap B_y} \mathrm{d}\mu \qquad L(x,y) = \int_{C_x \cap D_y} \mathrm{d}\nu$$

$$F_1(x) = \int_{A_x} \mathrm{d}\mu \qquad L_1(x) = \int_{C_x} \mathrm{d}\nu$$

$$F_2(y) = \int_{B_y} \mathrm{d}\mu \qquad L_2(y) = \int_{D_y} \mathrm{d}\nu$$

Probability copula and Lévy copula, compared

$$C(u,v) = F\big(F_1^{-1}(u), F_2^{-1}(v)\big) \qquad K(u,v) = L\big(L_1^{-1}(u), L_2^{-1}(v)\big)$$

$$\text{or } F(x,y) = C\big(F_1(x), F_2(y)\big) \qquad \text{or } L(x,y) = K\big(L_1(x), L_2(y)\big)$$

The uniform margin condition …

$$C_1(u) = C(u,1) = u \qquad K_1(u) = K(u,\infty) = u$$

$$C_2(v) = C(1,v) = v \qquad K_2(v) = K(\infty,v) = v$$

and the grounded condition …

$$C(0,v) = C(u,0) = 0 \qquad K(0,v) = K(u,0) = 0$$

Probability density and Lévy density, compared

If the probability measure and the Lévy measure have densities $f(x, y)$ and $l(x, y)$, then so also do their copulas, as here.

$$c(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v) = \frac{\partial^2}{\partial x \partial y} F(x, y) \Bigg/ \left(\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} \right) = \frac{f(x, y)}{f_1(x) f_2(y)}$$

$$k(u, v) = \frac{\partial^2}{\partial u \partial v} K(u, v) = \frac{\partial^2}{\partial x \partial y} L(x, y) \Bigg/ \left(\frac{\partial L_1}{\partial x} \frac{\partial L_2}{\partial y} \right) = \frac{l(x, y)}{l_1(x) l_2(y)}$$

Special copulas

The probability copula class and Lévy copula class contain various special copulas. Among them are the Fréchet-Hoeffding upper and lower limit copulas, and the independent copulas, as here.

$$C_{\uparrow}(u,v) = \min(u,v) \qquad K_{\uparrow} = \min(u,v)$$
$$C_{\downarrow}(u,v) = \max(u+v-1,0) \qquad K_{\downarrow} = \text{(no analogue)}$$
$$C_{\perp}(u,v) = uv \qquad K_{\perp}(u,v) = u \cdot \mathbb{1}_{\{v=\infty\}} + v \cdot \mathbb{1}_{\{u=\infty\}}$$

For a good foundation in Lévy copulas see (Applebaum 2004) and (Kallsen and Tankov 2006).

I. Subprime loan defaults and economic decline

An issue of public policy

What is the relationship between X_i and Y_i?

Let X_i be the fraction of defaults in period i, and Y_i be the negative of the growth rate (the decline rate) of the Gross National Product.

$$X_i = \frac{D_i}{T_i} \qquad\qquad Y_i = \log\left(\frac{G_{i-1}}{G_i}\right)$$

where $D_i \geqslant 0$ and $T_i > 0$, so $X_i \in [0,1]$, and where $G_i > 0$, so $Y_i \in \mathbb{R}$.

For a study on the distribution of loan losses see (Vasicek 1991). For a dynamic extension of Vasicek's model see (Lamb and Perraudin 2006). For an examination of the causes of corporate default clustering see (Das, Duffie, Kapadia, and Saita 2007).

Choosing a copula

Gumbel good, maybe rotated Clayton better — probably not Frank.

- Gumbel: Has strong right tail dependence, as expected from anecdotal evidence.
- Rotated Clayton: With similar features to Gumbel, has stronger right tail dependence.
- Frank: Exhibits symmetric dependence, so may not explain well.

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A question —

Given such a choice, how best does one provide a fit?

Fitting a copula

To keep in mind —

Better methods of fitting a copula will make all of copula theory more useful.

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Some approaches to parameter estimation —

1. Least squares comparison to independent copula
2. Maximum likelihood
3. Conditional probability integral transform (CPIT) (Berg and Bakken 2007)

The CPIT method . . .

CPIT allows for explicit weighting of observations

in an empirical copula near the boundaries of its domain allowing for more robust fitting of dependent tails.

The transform identifies the independent uniform $[0, 1]$ variates associated with a random draw from a known probability distribution. It is the inverse of making such a draw starting with the variates.

…relies on this definition.

If $Z = (X, Y)$ be a random variable on $(\mathbb{R}^2, \mathcal{B}, \mathbf{P})$, with distribution $F(x, y)$ having margins $F_1(x)$ and $F_2(y)$, then

$$\text{cpit}(F) := \big(F_1(x), F_{2|1}(y|x)\big),$$
$$\text{where } F_{2|1}(y|x) = \left.\frac{\partial F}{\partial x}\right|_{(x,y)}$$

Observe cpit(F) is uniform on $[0, 1]^2$.

II. Father and son accident propensity

An issue of joint responsibility

Again, what is the relationship between X_i and Y_i?

Consider the set of fathers with sons, or of mothers and daughters, each of whom drives. Let X_i and Y_i, respectively, be the fractions of those parents and children having accidents in period i.

$$X_i = \frac{P_i}{T_i} \qquad Y_i = \frac{C_i}{T_i}$$

where $0 \leqslant P_i, C_i \leqslant T_i > 0$, so $X_i, Y_i \in [0, 1]$.

Choosing a copula

Frank may be good — probably not Gumbel or Clayton, either direct or rotated.

- Frank: Has strong right and left tail dependence, as expected from anecdotal evidence.
- Gumbel and Clayton: Exhibit asymmetric dependence, so may not explain well.

Note that this example compares to Francis Galton's original study defining regression by utilizing Adrien-Marie Legendre's concept of least squares. See (Galton 1886) and (Legendre 1805).

Symmetric role of variables

Comparing, as we just have, a copula study to a regression study, we note these points.

- The regression study has an explanatory variable (father's height) and an explained variable (son's height.)
- The copula study has variables entering symmetrically (father's and son's driving records.) This is not to say that the resulting copula is symmetric. It may be (Archimedean, *e.g.*)

The risk manager's credo

Know what the risk is and how to measure it

before trying to manage it.

The nature of risk

Not long ago *one* concept of financial risk dominated all others. This was *variability of returns* and related proxies such as relative covariability to that of a broad market index, as captured in the now legendary “beta value.”

$$\beta = \frac{\mathrm{cov}(Y, X)}{\sigma^2(X)}$$

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$$\beta = \frac{\mathrm{cov}(Y, X)}{\sigma^2(X)}$$

Three well-respected investigators — Harry M. Markowitz, Merton H. Miller, and William F. Sharpe — won Nobel Prizes in Economics in 1990, all assuming throughout the works cited in their awards, this fundamental concept of risk.

Assumptions too simple

A problem ensued, however.

Along with the one concept of risk came the convenient assumption that the logarithms of returns are normally distributed.

All dependency relationships among market-related random variables, therefore, came to the fore as a simple, invariant object — the covariance matrix.

Some facts to consider

Before the ink was dry (much before!) on those Nobel Prize certificates, three things were well understood in the academic community.

1. Distributions of returns may not be — in fact are not — multinormal.
2. Variances and covariances may not even exist, *e.g.*, α-stable distributions.
3. Any indicators of return dispersion may well — in fact typically do — change through time.

Coherent and convex risk measures

Given the inadequacy of historic risk concepts to serve the needs of modern analysis and application researchers have turned to new ideas and methods. Among the recent studies to emerge exploring the foundations of risk and its measurement are these.

1. (Artzner, Delbaen, Eber, and Heath 1999)
   Defines and examines properties of coherent risk measures.
2. (Frittelli and Rosazza Gianin 2005)
   Studies law invariant coherent and convex risk measures.
3. (Mataramvura and Øksendal 2005)
   Researches convex risk measures in stochastic differential games.
4. (Rosazza Gianin 2006)
   Investigates on g-expectations and convex risk measures.

A forward look

Risk measures relate directly to premiums for insurance

and to prices of financial derivative instruments. As such, understanding risk leads to a more efficient economy.

Not understanding risk requires prudently the maintenance of expensive, otherwise unused reserves to prepare for unanticipated contingencies. This phenomenon itself warrants academic inquiry as an aspect of decision making with only partial information.

Copular convolution and spatial relations

$$
\begin{array}{ccccc}
(X & \xrightarrow[\text{Random Variables}]{\text{Joint}} & Y) & \longrightarrow & X+Y \\
\downarrow & & \downarrow & & \downarrow \text{ if independent} \\
(F & \xrightarrow[\text{Distributions}]{\text{Joint}} & G) & \longrightarrow & F\bigstar G \\
\hline\hline
(F_1 & \xrightarrow[\text{Margins}]{\text{First}} & G_1) & \longrightarrow & F_1\bigstar G_1 \\
\downarrow & & \downarrow & & \downarrow \text{ if independent} \\
(C & \xrightarrow{\text{Copulas}} & D) & \xrightarrow{\textit{Definition}} & C\widehat{\bigstar}D \\
\uparrow & & \uparrow & & \uparrow \text{ if independent} \\
(F_2 & \xrightarrow[\text{Margins}]{\text{Second}} & G_2) & \longrightarrow & F_2\bigstar G_2
\end{array}
$$

Copular evolution and temporal relationships

$$
\begin{array}{ccccc}
(X_s & \xrightarrow[\text{Random Variables}]{\text{Joint}} & X_t) & \longrightarrow & X_{s+t} \\
\downarrow & & \downarrow & & \downarrow \text{ if independent} \\
(F_s & \xrightarrow[\text{Distributions}]{\text{Joint}} & F_t) & \xrightarrow{\textit{Definition}} & {\color{red} F_s \,\widetilde{\bigstar}\, F_t} \\
\hline\hline
(F_{1s} & \xrightarrow[\text{Margins}]{\text{First}} & F_{1t}) & \longrightarrow & F_{1s} \bigstar F_{1t} \\
\downarrow & & \downarrow & & \downarrow \text{ if independent} \\
(C_s & \xrightarrow{\text{Copulas}} & C_t) & \xrightarrow{\textit{Definition}} & {\color{red} C_s \,\widehat{\bigstar}\, C_t} \\
\uparrow & & \uparrow & & \uparrow \text{ if independent} \\
(F_{2s} & \xrightarrow[\text{Margins}]{\text{Second}} & F_{2t}) & \longrightarrow & F_{2s} \bigstar F_{2t}
\end{array}
$$

Liberating copulas
sequences in space and time

Random variables do not have copulas, except indirectly.

Distributions have copulas.

Liberating copulas
sequences in space and time

Random variables do not have copulas, except indirectly.

Distributions have copulas.

Sequences of copulas arise naturally in stochastic analysis,

be they sequences of ordinary copulas, or sequences of Lévy copulas. Consider first a sequence of ordinary copulas in space, $\{C_t(i)\}$, or in time, $\{C_i(t)\}, i \in \mathrm{N}, t \geqslant 0$, such as

- $C_t(i)$, the copula linking $X_t(i)$ and $X_t(i-1)$ in a sequence of stochastic processes $\{X(i)\}$, or
- $C_i(t)$, the copula linking $X_i(t)$ and $X_i(t-1)$ in stochastic process X_i.

Lévy copula sequences and limits

Sequences of Lévy copulas, as relating to Lévy processes, can only occur nontrivially within sequences of multidimensional processes; within a single multidimensional process the Lévy copula, as the Lévy measure, remains invariant.

Therefore, consider as example the sequence of copulas $L(i)$

of the processes $Y(i), i \in \mathrm{N}$. Questions arise.

- Does $L(i)$ converge, or does a subsequence converge? (Recall that Lévy copulas are in general unbounded.)
- If convergent to $\overline{L}$, is $\overline{L}$ of a special type, like the independent copula (supported on the axes)?

Dependent volatility and jump measure

Have you heard? The market is jumpy today.

Now consider the generalized Itô-Lévy process X_t satisfying the stochastic integral equation

$$X_t = X_0 + \int_0^t \alpha(s, X_s, \omega)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, \omega)\,\mathrm{d}B_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X_s, z, \omega)\,\widetilde{N}(\mathrm{d}s, \mathrm{d}z),$$

for which the differential form is

$$\mathrm{d}X_t = \alpha(t, X_t, \omega)\,\mathrm{d}t + \sigma(t, X_t, \omega)\,\mathrm{d}B_t + \int_{\mathbb{R}_0} \gamma(t, X_t, z, \omega)\,\widetilde{N}(\mathrm{d}t, \mathrm{d}z)$$

A driven process

The presence of X_s in this integral

(X_t in the differential form) provides a dependence structure among the drift term, the Brownian term, and the jump term.

One could consider any of these appearances of X_s as a driven process, perhaps subordinated, applying its influence both to volatility and to the jump measure.

Of particular interest is the relationship between the last two terms.

A copula for volatility and jump measure

All that is necessary to create a copula

between volatility and a jump measure is to have a joint probability distribution for them.

Such a distribution is implicit in the given formulation, and could be subject to specific closed-form evaluation in special cases, numeric computation in others.

These copulas could be new, and of special research interest.

Co-integration

In 2003 Clive Granger won the Nobel Prize in Economics

for his seminal contributions to the theory of co-integration, which he and Robert Engle invented and advanced (Engle and Granger 1987). This theory relates how non-stationary series, when differenced, can in linear combination, be stationary. If so, the series are said to be "co-integrated." A vector of weights producing stationarity is said to be a "co-integrating vector."

Well, clearly co-integrated series exhibit dependencies. An interesting research project would be to compare co-integration concepts of dependency with copular concepts of dependency, both in theory and practice.

Recent contributions in risk management

Here is a selection of articles impacting on dependency in risk management.

1. (Vasicek 1991)
   Defines a probability of loss function on a loan portfolio.
2. (Schönbucher and Schubert 2001)
   Studies an approach to incorporate dynamic default dependency.
3. (Embrechts, McNeil, and Lindskog 2003)
   Researches uses of copulas in integrated risk management.
4. (Frey and Backhaus 2004)
   Investigates portfolio credit risk with interacting default intensities.

Terrorism

- Just another natural disaster?
- The twin curses
  - Dimensionality
  - Small samples

Industrialization and global warming

- To what extent related?
- The variables
  - Industrial activity
  - Carbon dioxide

Over-extension of credit and bank failures

- To what extent related?
- The variables
  - Proliferation of incomprehensible debt instruments
  - Defaults on payments of interest or principal

Recent contributions in finance

Here is a selection of articles impacting on dependency in finance.

1. (Duffie and Singleton 1999)
   Defines models of contingent claims with default risk.
2. (Meyer-Brandis and Proske 2004)
   Studies optimal filtering to estimate Lévy measures having time-inhomogeneous densities.
3. (Malo and Kanto 2006)
   Researches multivariate GARCH models for dynamic hedging in electricity markets.
4. (Ta Thi Kieu and Øksendal 2007)
   Investigates a maximum principle for stochastic differential games in the context of partial information.

Micro-lending and weather

In 2006 Muhammad Yunus and Grameen Bank,

which he founded, won the Nobel Peace Prize. Is such a business vulnerable to the weather, specifically monsoons? If there is a relationship, what is it? Can copula theory help to analyze such a risk, or have anything to say about how to hedge it?

- Concentration in regions
- Vulnerability of borrowers
  - Collateral
  - Jobs

Project stress — Airbus A380

Large companies have dependent risk exposures

for similarly situated products. What are the concordant risks? Are they hedgeable? Can copulas help in reasoning?

- Companies and products
- Internal dependence — A380 and A350
  - Common physical structures
  - Cost substitution

Real options and optimal exercise

Companies face sets of choices

for implementing competing projects. What are the concordant risks? Are they hedgeable? Can copulas help in reasoning?

- Synergy/Antergy of contemporary projects
- Serial/Parallel dependence
    - Initiate one, then the other, or in reverse.
    - Initiate together.

Epilogue

It doesn't work to leap a ten foot chasm in two five foot jumps.
— American proverb

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