INTEGER POINTS OF NONLINEAR MANIFOLDS

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Preface

I Introduction

The goal of this study is to initiate a theory of systematic recursive searches on algebraic varieties for integer point solutions. Symbolically, if a variety be given by the real locus of a finite set of polynomial equations with the integral coefficients,

$$\{p_i(x) = p_i(x_0, x_1, \ldots, x_n) = 0\}, \tag{P-1}$$

the task is to find and analyze transformations $\{T_j\}$ such that if $x^0 = \left(x_0^0, x_1^0, \ldots, x_n^0\right)$ is a solution in integers to Equation (P-1), then so are the $\left\{T_j^i(x^0)\right\}$. Motivation derives from nonlinear programming problems which give Equation (P-1) as constraints. Admittedly, the task is large; a complete solution necessarily acknowledges proof of Fermat's famous "Last Theorem," rendering $x_0^d + x_1^d - x_2^d = 0$ inadmissible as a constraint for $d > 2$. This paper therefore takes the more tractable approach of restricting attention to varieties specified by a single polynomial, with much of the analysis further focused on equations which are quadratic in pairs of their variables.

The study develops in four major chapters, numbered 1 through 4. Chapter 1 begins by viewing polynomials as covariant tensors, then proceeds to an analysis of integer point simplexes, the functions of which are isomorphic to polynomials. Principal thought then turns to an investigation of several important cardinality functions defined on the space of simplexes. Chapter 2 continues by discussing in detail the integer solutions to a specific heterogeneous binary quadratic polynomial generated in Chapter 1. Insights gained by this solution lead in Chapter 3 to an expanded theory of arbitrary heterogeneous binary quadratics, *i.e.*, to equations of the form,

$$ay^2 + byx + cx^2 + dy + ex + f = 0 \tag{P-2}$$

Chapter 4 shows these techniques to be applicable to searches of higher dimensional varieties which are pairwise quadratic in their variables.

II Literature review

The literature of recursive solution generation to polynomial forms is sparse. Two classical references, however are noteworthy. In 1883, M. S. Réalis writing in (Réalis 1883) demonstrated that if $\binom{p_n}{q_n}$ solves

$$(m+4)p^2 - mq^2 = 4, \tag{P-3}$$

then so does

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (m+2) & m \\ (m+4) & (m+2) \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

He stated (but did not prove) that the above transform and its powers applied to the starting solutions $\binom{\pm 1}{\pm 1}$ provide a complete solution set. The Equation (P-3), with $m = 1$, plays the fundamental role in Chapter 2, and also appears in Chapter 3, where a completeness proof is given.

The other reference is to an article of M. Rignaux appearing in (Rignaux 1919). Therein he stated that the quadratic Equation (P-2) with integer coefficients is solved for an indeterminate integer μ by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mu \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} - \begin{pmatrix} \mu - 2 \\ \Delta \end{pmatrix} \begin{pmatrix} 2ae - bd \\ 2cd - be \end{pmatrix}, \tag{P-4}$$

where $\Delta = b^2 - 4ac > 0$, and $\binom{x_n}{y_n}$ and $\binom{x_{n-1}}{y_{n-1}}$ are other solutions. A counterexample to this long held "theorem" appears as Example 4 of Chapter 3.

In reviewing the relevant literature, one should note that there are myriad references to equations of the type Equation (P-2), especially for the homogeneous case $d = e = f = 0$, but virtually all only characterize the solutions, and do not discuss recursive generation. In this genre are the classical papers of Euler, Lagrange, Legendre, Gauss, and others, as surveyed by L. E. Dickson in his monumental narrative bibliography (Dickson 1919; Dickson 1920; Dickson 1923). A typical result of the Eighteenth Century investigations is the identification of all rational solutions, from which it is observed that one may select all integral solutions.

For the reader interested in the more modern literature, examine first the bibliographies of the Encyclopædia Britannica[1] under "Number Theory," "Diophantine Analysis," and "Tensor Algebra." For recent published papers of general interest, consult Mathematical Reviews. For review of algebraic varieties and polynomial rings, see (Birkhoff and MacLane 1953; Zariski and Samuel 1958); for review of tensor analysis see (Nickerson, Spencer, and Steenrod 1959; Nelson 1967).

III Acknowledgments

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[1]This reference should not be discounted by the serious student, as most mathematically oriented entries are authoritatively written.

the Graduate School of Business Administration, University of California, Berkeley. The work has been amended and revised during the intervening years, with the current version having been prepared during December 2007, while the author was in residence at the Centre of Mathematics for Applications and Department of Mathematics, University of Oslo. The author gratefully acknowledges the considerable assistance of these institutions, and their faculty and staff, in facilitating this research.

Chapter 1

Integer simplexes

1 Tensor representations and the simplex domains

Consider the polynomial of degree d in $n+1$ variables,

$$p(x_0, x_1, \ldots, x_n) \tag{1.1}$$

Equivalently, this form is

$$A^{(d)}x^d + A^{(d-1)}x^{d-1} + \cdots + A^{(0)}, \tag{1.2}$$

where $A^{(k)}x^k \in P^{(k)}$, the homogeneous polynomials of degree k, means

$$\sum_{i_0=0}^{n}\sum_{i_1=0}^{n}\cdots\sum_{i_{k-1}=0}^{n} a_{i_0 i_1 \cdots i_{k-1}} x_{i_0} x_{i_1} \cdots x_{i_{k-1}}, \tag{1.3}$$

for suitable choice of coefficients. In this format, the polynomial is a covariant (or dual) tensor (Nelson 1967). Owing to the commutativity of the variables, one may change the terms of Equation (P-2) in any way consistent with the invariance of $\sum_{d\pi} a_{i_0 i_1 \cdots i_{k-1}}$, taken over the set $d\pi$ of all *distinct* permutations of the indices. In particular, one may symmetrize $A^{(k)}$ to $B^{(k)}$, written $B^{(k)} = \text{sym}\left(A^{(k)}\right)$, by setting $b_{i_0 i_1 \cdots i_{k-1}} = 1/k! \cdot \sum_{\pi} a_{i_0 i_1 \cdots i_{k-1}}$, taken over the set π of *all* permutations of the indices, or *pack* $A^{(k)}$ to $C^{(k)}$, written $C^{(k)} = \text{pack}\left(A^{(k)}\right)$, by $c_{i_0 i_1 \cdots i_{k-1}} = \sum_{d\pi} a_{i_0 i_1 \cdots i_{k-1}}$, taken over $d\pi$ if $i_0 \leqslant i_1 \leqslant \cdots \leqslant i_{k-1}$, or by setting $c_{i_0 i_1 \cdots i_{k-1}} = 0$, otherwise.

If one takes either symmetric or packed representations for all polynomials, and if the 'sym' or 'pack' operators, respectively, follow normal tensor multiplication, then the resulting structures are *symmetric* or *packed* algebras. In either case it is only necessary to maintain $a_{i_0 i_1 \cdots i_{k-1}}$ for $i_0 \leqslant i_1 \leqslant \cdots \leqslant i_{k-1}$, because from these values one may construct the others. The packed representation is used exclusively henceforth, because Equation (1.3) becomes

$$\sum_{i_0=0}^{n}\sum_{i_1=0}^{n}\cdots\sum_{i_{k-1}=0}^{n} c_{i_0 i_1 \cdots i_{k-1}} x_{i_0} x_{i_1} \cdots x_{i_{k-1}}, \tag{1.4}$$

without necessity for expressions involving factorials when calculating products.

The following example concerning a multilinear composition operator serves to illustrate computation with packed tensors.

Let $P^{(k)}$ now be the packed tensor representations of homogeneous polynomials of degree k. One may define a multilinear transformation,

$$L : \underbrace{P^{(k)} \times P^{(k)} \times \cdots \times P^{(k)}}_{k \text{ factors}} \mapsto P^{(k)}$$

by

$$B^{(k)}x^k = L\left({}^0A^{(k)}x^k, {}^1A^{(k)}x^k, \ldots, {}^{k-1}A^{(k)}x^k\right), \tag{1.5}$$

where

$$b_{i_0 i_1 \cdots i_{k-1}} = \sum_{j_1=0}^{n} \sum_{j_2=j_1}^{n} \cdots \sum_{j_{k-1}=j_{k-2}}^{n} {}^0a_{\underline{i_0} j_1 \cdots j_{k-1}} \, {}^1a_{j_1 \underline{i_1} j_2 \cdots j_{k-1}} \cdots {}^{k-1}a_{j_1 j_2 \cdots j_{k-1} \underline{i_{k-1}}} \tag{1.6}$$

If $k = 2$, Equation (1.6) reduces to the usual definition of matrix multiplication, so that if one rewrite $B^{(2)}x^2$, ${}^0A^{(2)}x^2$, ${}^1A^{(2)}x^2$ as $x'Bx$, $x'{}^0Ax$, $x'{}^1Ax$ for matrices B, ${}^0A$, ${}^1A$, (where prime indicates transposition to row vector,) then Equation (1.5) becomes

$$x'(B)x = L\left(x'\,{}^0Ax, x'\,{}^1Ax\right) = x'\left({}^0A\,{}^1A\right)x \tag{1.7}$$

It is not necessary to pack the tensor developed according to Equation (1.6), as the result is already packed if the arguments are. To observe this fact, note that if $i_m > i_n$ for some $m < n$, then any sequence $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_{k-1}$ which admits i_m thus: $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_m \leqslant i_m \leqslant j_{m+1} \leqslant \cdots \leqslant j_{k-1}$, necessarily fails to be monotone for i_n, thus: $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_{m+1} \leqslant \cdots \leqslant j_n > i_n$, rendering the intermediate product zero.

Continuing from Equation (1.4), one sees that the coefficients $\left\{c_{i_0 i_1 \cdots i_{k-1}}\right\}$ exist only for $0 \leqslant i_0 \leqslant i_1 \leqslant \cdots \leqslant i_{k-1} \leqslant n$. This domain in $J^k = J \times J \times \cdots \times J$, with k factors and J symbolizing the integers, is an *integer point simplex*. As presented, two parameters determine a simplex — the dimension k and the limit index number n. To preserve mnemonic significance in the sequel, the dimension of the simplex (the degree of the related homogeneous polynomial) is d, and the domain is therefore a (d, n) simplex. As this theory unfolds, a parallel theory relating to domains satisfying $0 < i_0 < i_1 < \cdots < i_{d-1} < n+1$, called *strict* (d, n) *simplexes*, is also included. Note that a strict (d, n) simplex becomes a $(d, n+1)$ simplex upon changing the relations from $<$ to $\leqslant$. Conversely, a (d, n) simplex becomes a strict $(d, n-1)$ simplex upon changing the relations from $\leqslant$ to $<$.

At certain times alternate formulations of the defining relations for both types of simplexes are useful. One such formulation evolves by applying the transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ -1 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & -1 & 1 & \cdots & \cdots & \cdots \\ \hdotsfor{6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix},$$

(that is, with 1's on the major diagonal, −1's on the subdiagonal, and zeros elsewhere) to the embedding space J^d.

If $I = (i_0, i_1, \ldots, i_{d-1})^T$ and $\widehat{I} = (\hat{i}_0, \hat{i}_1, \ldots, \hat{i}_{d-1})^T = TI$, then for the [non-strict] (d, n) simplexes, $\hat{i}_k \geqslant 0$ for each k, $0 \leqslant k \leqslant d-1$, and $\sum_{k=0}^{d-1} \hat{i}_k \leqslant n$; for the strict (d, n) simplexes, $\hat{i}_k > 0$ for each k, $0 \leqslant k \leqslant d-1$, and $\sum_{k=0}^{d-1} \hat{i}_k < n+1$. These alternate formulations are consistent with the usual specification of constraints in linear programs, to which one frequently applies "simplex" methods for solution. The transformation T is necessarily invertible to provide isomorphism.

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & \cdots & \cdots \\ \hdotsfor{6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

(that is, with 1's on the major diagonal and the lower triangle, and zeros elsewhere.)

2 Cardinality theorems and linearization

A first step in the study of simplexes is to determine cardinality, given the parameters d and n. The following five theorems establish the needed results.

Theorem 1.1. *If* $V(d, n)$ *is the number of points in a* (d, n) *simplex, then* $V(d, n) = V(d, n-1) + V(d-1, n)$.

Proof. The set $\{(i_0, i_1, \ldots, i_{d-1}) \mid 0 \leqslant i_0 \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} \leqslant n\}$ is the disjoint union of two sets. They are

$$\{(i_0, i_1, \ldots, i_{d-1}) \mid 0 \leqslant i_0 \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} \leqslant n-1\} \tag{1.8}$$

and

$$\{(i_0, i_1, \ldots, i_{d-1}) \mid 0 \leqslant i_0 \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} = n\} \tag{1.9}$$

The former has $V(d, n-1)$ points; the latter has $V(d-1, n)$ points. □

Theorem 1.2. *If* $W(d,n)$ *is the number of points in a strict* (d,n) *simplex, then* $W(d,n) = W(d,n-1) + W(d-1,n-1)$.

Proof. The set $\{(i_0, i_1, \ldots, i_{d-1}) \mid 0 < i_0 < i_1 < \cdots < i_{d-1} < n+1\}$ is the disjoint union of two sets. They are

$$\{(i_0, i_1, \ldots, i_{d-1}) \mid 0 < i_0 < i_1 < \cdots < i_{d-1} < n\} \tag{1.10}$$

and

$$\{(i_0, i_1, \ldots, i_{d-1}) \mid 0 < i_0 < i_1 < \cdots < i_{d-1} = n\} \tag{1.11}$$

The former has $W(d,n-1)$ points; the latter has $W(d-1,n-1)$ points. □

Theorem 1.3. $V(d,n) = \binom{d+n}{d} = \binom{d+n}{n}$

Proof. By induction

$$V(1,n) = \left\{(i_0) \mid 0 \leqslant i_0 \leqslant n\right\} = n+1 = \binom{1+n}{1}$$
$$V(d,0) = \left\{(i_0, i_1, \ldots, i_{d-1}) \mid 0 \leqslant i_0 \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} \leqslant 0\right\} = 1 = \binom{d+0}{0}$$

By Theorem 1.1,

$$\begin{aligned} V(d,n) &= V(d,n-1) + V(d-1,n) \\ &= \binom{d+n-1}{d} + \binom{d-1+n}{d-1} \\ &= \frac{(d+1-n)!}{d!(n-1)!} + \frac{(d-1+n)!}{(d-1)!n!} \\ &= \frac{n(d+n-1)!}{d!n!} + \frac{d(d-1+n)!}{d!n!} \\ &= \frac{(d+n)(d+n-1)!}{d!n!} \\ &= \frac{(d+n)!}{d!n!} \\ &= \binom{d+n}{d} = \binom{d+n}{n} \end{aligned}$$

□

One may now define consistently, that

$$V(0,n) = \binom{0+n}{n} = 1$$

Theorem 1.4. $W(d, n) = \binom{n}{d} = \binom{n}{n-d}$

Proof. Simply observe that a point of the strict (d, n) simplex is a choice of d distinct indices from a collection of n indices. □

At this point one can relate the function $V(d, n)$ and $W(d, n)$.

Theorem 1.5. $W(d, n) = V(d, n - d)$ *and* $V(d, n) = W(d, n + d)$

Proof. Omitted

Portions of both functions on their respective domains appear below.
V, defined on $\{(d, n) \mid d \geqslant 0, n \geqslant 0\}$:

$$\begin{bmatrix} 1 & 6 & 21 & 56 & 126 & 252 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

W, defined on $\{(d, n) \mid d \geqslant 0, n \geqslant d\}$:

$$\begin{bmatrix} 1 & 5 & 10 & 10 & 5 & 1 \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 3 & 3 & 1 & & \\ 1 & 2 & 1 & & & \\ 1 & 1 & & & & \\ 1 & & & & & \end{bmatrix}$$

The next step of the analysis is to examine functions of the class $\{\phi_j(i_0, i_1, \dots, i_{d-1})\}$, which assigns a unique integer to each point of a simplex (of either type.) If such a function is monotone on the indices $\{(i_0, i_1, \dots, i_{d-1})\}$ considered as numbers in base $n + 1$, *i.e.*, as

$$\left\{ \sum_{k=0}^{d-1} i_k (n+1)^{d-1+k} \right\},$$

and has range included by $[0, V(d, n) - 1]$ or $[0, W(d, n) - 1]$, then the function is a *mapping function*, and its associated polynomial

$$p(x_0, x_1, \dots, x_n) = \sum_{d\pi} \left(\phi(i_0, i_1, \dots, i_{d-1}) \cdot x_{i_0} x_{i_1} \cdots x_{i_{d-1}} \right),$$

taken over all points of the simplex, is its *mapping variety*. The process of assigning a mapping function is *linearization*. Functions F_n^d and G_n^d linearizing both types of simplexes are given below.

I	$F_3^3(I)$
(0,0,0)	19
(0,0,1)	18
(0,0,2)	17
(0,0,3)	16
(0,1,1)	15
(0,1,2)	14
(0,1,3)	13
(0,2,2)	12
(0,2,3)	11
(0,3,3)	10
(1,1,1)	9
(1,1,2)	8
(1,1,3)	7
(1,2,2)	6
(1,2,3)	5
(1,3,3)	4
(2,2,2)	3
(2,2,3)	2
(2,3,3)	1
(3,3,3)	0

Table 1.1: Example: $F_3^3(I)$

First consider the non-strict case. Let $I^0 = \left(i_0^0, i_1^0, \ldots, i_{d-1}^0\right)^T$. The plan is to count points at higher values than I^0, as base $(n+1)$ numbers. If $F_n^d(I^0)$ be the desired function, then $F_n^d(I^0)$ is the sum of cardinalities of the sets $\{S_k\}$ for $0 \leqslant k \leqslant d-1$, where S_k is the $\left[d-k, n-\left(i_k^0+1\right)\right]$ simplex

$$\left\{(i_0, i_1, \ldots, i_{d-1}) \mid i_k^0 + 1 \leqslant i_k \leqslant i_{k+1} \leqslant \cdots \leqslant i_{d-1} \leqslant n\right\}$$

Dropping index superscripts and simplifying,

$$V_n^d(I) = \sum_{k=0}^{d-1} V(d-k, n-i_k-1)$$

It follows readily that $F_n^d(0,0,\ldots,0) = V(d,n) - 1$. and that $F_n^d(n,n,\ldots,n) = 0$. An example of the $(3,3)$ simplex appears in Table 1.1.

Next, consider the strict case. Let I^0 be as before. If G_n^d be the desired function, then $G_n^d(I^0)$ is the sum of cardinalities of the sets $\{T_k\}$ for $0 \leqslant k \leqslant d-1$, where T_k is the strict $\left(d-k, n-i_k^0\right)$ simplex

$$\left\{(i_0, i_1, \ldots, i_{d-1}) \mid i_k^0 < i_k < i_{k+1} < \cdots < i_{d-1} < n+1\right\}$$

I	$G_6^4(I)$
(1,2,3,4)	14
(1,2,3,5)	13
(1,2,3,6)	12
(1,2,4,5)	11
(1,2,4,6)	10
(1,2,5,6)	9
(1,3,4,5)	8
(1,3,4,6)	7
(1,3,5,6)	6
(1,4,5,6)	5
(2,3,4,5)	4
(2,3,4,6)	3
(2,3,5,6)	2
(2,4,5,6)	1
(3,4,5,6)	0

Table 1.2: Example: $G_6^4(I)$

Dropping index superscripts and simplifying, $G_n^d(1, 2, \ldots, d) = W(d, n) - 1$, and that $G_n^d(n - d + 1, n - d + 2, \ldots, n) = 0$. An example for the strict $(4, 6)$ simplex appears in Table 1.2.

If one desires repeated computation of F_n^d and G_n^d for some given simplex (as for digital computer,) then the advance preparation of tables for lookup can be helpful. For F_n^d, each evaluation requires the summation of d terms, each of which can take $n + 1$ values. The resulting table is therefore necessarily of $d(n + 1)$ elements. For G_n^d, similar comments apply, except that $k < i_k < n - d + k + 2$. These restrictions remove from consideration two triangles (strict $(2, d)$ simplexes) from the full d by n table. Hence the total needed entries is

$$dn - 2\binom{d}{2} = dn - d(d - 1) = dn - d^2 + d$$

Relationships between F_n^d and G_n^d establish readily, using Theorem 1.5.

Theorem 1.6.

$$G_n^d(I) = F_n^d\big(I + (d - 1, d - 2, \ldots, 0)\big)$$

and

$$F_n^d(I) = G_n^d\big(I - (d - 1, d - 2, \ldots, 0)\big)$$

Proof. In the former case,

$$\begin{aligned} G_n^d(I) &= \sum_{k=0}^{d-1} W(d-k, n-i_k) \\ &= \sum_{k=0}^{d-1} V\big(d-k, n-(i_k+d-k-1)-1\big) \\ &= F_n^d\big(I+(d-1, d-2, \ldots, 0)\big) \end{aligned}$$

In the latter case,

$$\begin{aligned} F_n^d(I) &= \sum_{k=0}^{d-1} V(d-k, n-i_k-1) \\ &= \sum_{k=0}^{d-1} W\big(d-k, n-(i_k-d+k+1)\big) \\ &= G_n^d\big(I-(d-1, d-2, \ldots, 0)\big) \end{aligned}$$

□

3 The simplex overflow distribution

In the case of non-strict (d, n) simplexes, $F_n^d(I)$ for any I such that $0 \leqslant i_k \leqslant n$ for all k, $0 \leqslant k \leqslant d-1$, is within the bounds $[0, V(d, n)-1]$. This fact follows, as $F_n^d(I)$ is monotone on any of its components separately, and both $(0, 0, \ldots, 0)$ and $(n, n, \ldots, n)$ are in the simplex. Therefore, by counting frequencies of F_n^d over the *entire* hypercube

$$\left\{(i_0, i_1, \ldots, i_{d-1}) \;\middle|\; 0 \leqslant i_k \leqslant n, \text{ for all } k,\ 0 \leqslant k \leqslant d-1\right\},$$

one obtains a probability distribution on $[0, V(d, n)-1]$ called the *simplex overflow distribution*. An example for the $(3, 3)$ simplex appears in Table 1.3. Similar constructions for strict (d, n) simplexes do not exist, since some points $G_n^d(I)$ are out of range.

4 Inversion of linearization

One can reconstruct the index vector I easily if the function value $F_n^d(I)$ or $G_n^d(I)$ is known. The process is *inversion*, and is straightforwardly a reversal of the construction of F_n^d and G_n^d. The method is to subtract successively from the function value the cardinalities of the largest contained simplexes of dimensions $d, d-1, \ldots, 1$. The sequential determination of $i_0, i_1, \ldots, i_{d-1}$ is a byproduct of this exercise.

In more detail, starting with F_n^d, the algorithms are: y

(a) Take $F_n^d(I)$. Call this quantity the *linear residual* (lr). The lr is decremented upon the determination of each new index component.

$F_3^3(I)$	Frequency
19	1
18	1
17	1
16	2
15	1
14	2
13	4
12	3
11	3
10	4
9	3
8	4
7	6
6	5
5	5
4	6
3	5
2	4
1	3
0	1

Table 1.3: Overflow Distribution for $(3,3)$ simplex. Total points $64 = (n+1)^d = 4^3$

(b) Set a counter k to zero.

(c) Continue if $k < d$, else exit.

(d) Find the least integer i_k such that $V(d-k, n-i_k-1) \leqslant lr$.

(e) Decrement lr by $V(d-k, n-i_k-1)$.

(f) Repeat with step (c).

Step (d) presents certain special problems. One way to determine i_k (at least in the abstract) is first to extend $V(d-k, n-i_k-1)$ to a monotone continuous function V_c on i_k considered as an integer. Then $i_k = \lceil V_c^{-1}(lr) \rceil$. One continuous function to use is

$$\frac{(d-k+n-i_k-1)(d-k+n-i_k-2)\cdots(n-i_k)}{(d-k)!} = \frac{\Gamma(d-k+n-i_k)}{\Gamma(d-k+1)\,\Gamma(n-i_k)}$$

If the algorithm were coded for digital computer, roundoff error could cause problems, insofar as integer results are needed. An alternative scheme, which stays with the integer domain, is to make an initial trial with the maximum possible $i_k = n$. Let this trial be $t_0 = (d-k+j-1)/(j-1) \cdot t_{j-1}$.

For the case G_n^d, the algorithm is the same, with the following exceptions: Step (d): Find the least integer i_k such that $W(d-k, n-i_k) \leqslant lr$. Step (e): Decrement lr by $W(d-k, n-i_k)$. The similar comment about continuous extension of $W(d-k, n-i_k)$ on i_k to W_c monotone, applies. Then $i_k = \lceil W_c^{-1}(lr) \rceil$. An available continuous function is

$$\frac{(n-i_k)(n-i_k-1)\cdots(n-i_k-d+k+1)}{(d-k)!} = \frac{\Gamma(n-i_k)}{\Gamma(d-k+1)\,\Gamma(n-i_k-d+k+1)}$$

For integer-only generation, the same recursion as before obtains.

$$\begin{aligned} t_0 &= W(d-k, d-k-1) &&= V(d-k, -1) &&= 0, \\ t_1 &= W(d-k, d-k) &&= V(d-k, 0) &&= 1, \quad \text{and} \\ t_j &= \frac{d-k+j-1}{j-1} \cdot t_{j-1} \end{aligned}$$

5 Discrete integration on the cardinality functions

The V and W functions allow various single and double discrete integrations (summations.) First, for V, one has this Theorem.

Theorem 1.7.

$$V(d,n) = \sum_{k=0}^{n} V(d-1,k) = \sum_{(i,j)\in S_{00}(d,n)} V(i,j),$$

where

$$S_{00}(d,n) = \big\{(i,j) \,\big|\, i = d-1,\ 0 \leqslant j \leqslant n\big\}$$

Proof. By induction

$$\begin{aligned} V(d,0) &= \sum_{k=0}^{0} V(d-1,k) = V(d-1,0) = 1 \\ V(d,n) &= V(d,n-1) + V(d-1,n) \quad \text{by Theorem 1.1} \\ &= \sum_{k=0}^{n-1} V(d-1,k) + V(d-1,n) \\ &= \sum_{k=0}^{n} V(d-1,k) \end{aligned}$$

□

Corollary 1.8.

$$\begin{aligned} V(d,n) &= \sum_{k=0}^{n} V(k,d-1) = \sum_{(i,j)\in S_{01}(d,n)} V(i,j), \quad \textit{where} \\ S_{01}(d,n) &= \big\{(i,j) \,\big|\, 0 \leqslant i \leqslant n,\ j = d-1\big\}\,; \\ V(d,n) &= \sum_{k=0}^{d} V(n-1,k) = \sum_{(i,j)\in S_{10}(d,n)} V(i,j), \quad \textit{where} \\ S_{10}(d,n) &= \big\{(i,j) \,\big|\, i = n-1,\ 0 \leqslant j \leqslant d\big\}\,; \\ V(d,n) &= \sum_{k=0}^{d} V(k,n-1) = \sum_{(i,j)\in S_{11}(d,n)} V(i,j), \quad \textit{where} \\ S_{11}(d,n) &= \big\{(i,j) \,\big|\, 0 \leqslant i \leqslant d,\ j = n-1\big\} \end{aligned}$$

Proof. Immediate from the Theorem, and from Theorem 1.3 □

Using Theorem 1.5 and the related linear transformations $A : (d,n) \to (d,n+d)$ and $A^{-1} : (d,n) \to (d,n-d)$, the corresponding results for W are these.

Corollary 1.9.

$$W(d,n) = \sum_{k=0}^{n-d} V(d-1,k) = \sum_{k=0}^{n-d} W(d-1,d-1+k) = \sum_{(i,j)\in T_{00}(d,n)} W(i,j),$$

where $T_{00}(d,n) = A[S_{00}(d,n)]$;

$$W(d,n) = \sum_{k=0}^{n-d} V(k, d-1) = \sum_{k=0}^{n-d} W(k, d-1+k) = \sum_{(i,j)\in T_{01}(d,n)} W(i,j),$$

where $T_{01}(d,n) = A[S_{01}(d,n)]$;

$$W(d,n) = \sum_{k=0}^{d} V(n-d-1, k) = \sum_{k=0}^{d} W(n-d-1, n-d-1+k) = \sum_{(i,j)\in T_{10}(d,n)} W(i,j),$$

where $T_{10}(d,n) = A[S_{10}(d,n)]$;

$$W(d,n) = \sum_{k=0}^{d} V(k, n-d-1) = \sum_{k=0}^{d} W(k, n-d-1+k) = \sum_{(i,j)\in T_{11}(d,n)} W(i,j),$$

where $T_{11}(d,n) = A[S_{11}(d,n)]$

Proof. Immediate from the substitutions indicated above □

Remark. Note additionally that

$$\sum_{k=0}^{d} W(k,d) = \sum_{k=0}^{d} \binom{d}{k} = 2^d$$

The symmetries of S_{pq} and T_{pq} are useful when considering double summations, and are therefore now formalized. First, let $\widehat{S}$ be defined as the *reflection* of S, if both are sets of integer ordered pairs, and if $\widehat{S} = \{(j,i) \mid (i,j) \in S\}$. Note that $\widehat{\widehat{S}} = S$.

Lemma 1.10. $S_{pq}(d,n) = \widehat{S}_{p\bar{q}}(d,n) = S_{\bar{p}q}(n,d)$, *where the bars on subscripts indicate binary complementation.*

Proof. Verification is easy from the definitions. □

Corollary 1.11. $T_{pq}(d,n) = \widehat{T}_{p\bar{q}}(d,n) = T_{\bar{p}q}(n,d)$

Proof. Apply A. □

With the four distinct sets $S_{pq}(d,n)$ it is formally possible to express sixteen double summations:

$$V(d,n) = \sum_{(i,j)\in S_{pq}(d,n)} \sum_{(k,l)\in S_{rs}(i,j)} V(k,l) = \sum_{(k,l)\in X_{pqrs}(d,n)} V(k,l)$$

These sixteen sets can, however, be partitioned into pairs of identical sets and pairs of reflections. Furthermore, certain of the sets have identical intersections with the domain of V.

Theorem 1.12.

$$X_{p\bar{q}\bar{r}s} = X_{pqrs} = \widehat{X}_{pqr\bar{s}}$$

Proof.

$$\sum_{(j,i)\in S_{p\bar{q}}(d,n)} \sum_{(k,l)\in S_{\bar{r}s}(j,i)} V(k,l) =$$
$$\sum_{(i,j)\in \widehat{S}_{p\bar{q}}(d,n)} \sum_{(k,l)\in S_{\bar{r}s}(j,i)} V(k,l) =$$
$$\sum_{(i,j)\in S_{pq}(d,n)} \sum_{(k,l)\in S_{rs}(i,j)} V(k,l) =$$
$$\sum_{(i,j)\in S_{pq}(d,n)} \sum_{(k,l)\in S_{r\bar{s}}(i,j)} V(l,k)$$

□

An enumeration of the sets $\{X_{pqrs}\}$ follows. Arrows on the left connect reflections. Elements marked by asterisks and elements pointed by arrows on the right refer to the following Theorem.

$$a \to X_{0000} = X_{0110}$$
$$a \to X_{0001} = X_{0111}$$
$$*b \to X_{0010} = X_{0100} \leftarrow e$$
$$*b \to X_{0011} = X_{0101} \leftarrow f$$
$$c \to X_{1000} = X_{1110}$$
$$c \to X_{1001} = X_{1111}$$
$$d \to X_{1010} = X_{1100} \leftarrow f*$$
$$d \to X_{1011} = X_{1101} \leftarrow e*$$

Theorem 1.13. *Let $\overline{V}$ be the domain of V. Then $X_{0011} \cap \overline{V} = X_{1100} \cap \overline{V}$.*

Proof.

$$\sum_{(i,j)\in X_{0011}(d,n)} V(i,j) = \sum_{k=0}^{n} \sum_{j=0}^{d-1} V(j,k-1)$$

from the definitions, but $V(j,-1)$ is not in $\overline{V}$. Hence,

$$\sum_{k=1}^{n} \sum_{j=0}^{d-1} V(j,k-1) = \sum_{k=0}^{n-1} \sum_{j=0}^{d-1} V(j,k) \tag{1.12}$$

is the restriction to $\overline{V}$. Similarly,

$$\sum_{(i,j)\in X_{1100}(d,n)} V(i,j) = \sum_{k=0}^{d}\sum_{j=0}^{n-1} V(k-1,j),$$

but $V(-1,j)$ is not in $\overline{V}$. Hence,

$$\sum_{k=1}^{d}\sum_{j=0}^{n-1} V(k-1,j) = \sum_{k=0}^{d-1}\sum_{j=0}^{n-1} V(k,j) \tag{1.13}$$

is the restriction to $\overline{V}$.

Interchanging the variables k and j in Equation (1.13) makes it identical to Equation (1.12). □

Corollary 1.14. $X_{0010} \cap \overline{V} = X_{1101} \cap \overline{V}$.

Proof.

$$X_{0010} = \widehat{X}_{0011} \qquad \text{and} \qquad X_{1101} = \widehat{X}_{1100}$$

by Theorem 1.12. Hence,

$$X_{0010} \cap \overline{V} = \widehat{X}_{0011} \cap \overline{V} \quad \text{and} \quad X_{1101} \cap \overline{V} = \widehat{X}_{1100} \cap \overline{V}$$

Since $\overline{V}$ is its own reflection, it follows that the right hand sides of the second line are equal. Therefore,

$$\begin{aligned}
&(i,j) \in \widehat{X}_{0011} \cap \overline{V} \iff \\
&(i,j) \in \widehat{X}_{0011} \cap \widehat{\overline{V}} \iff \\
&(i,j) \in \widehat{X}_{0011} \quad \text{and} \quad (i,j) \in \widehat{\overline{V}} \iff \\
&(j,i) \in X_{0011} \quad \text{and} \quad (j,i) \in \overline{V} \iff \\
&(j,i) \in X_{0011} \cap \overline{V}
\end{aligned}$$

Apply now the same reasoning to $\widehat{X}_{1100} \cap \overline{V}$. □

Truncation of the certain sets $\{X_{pqrs}\}$ as per the Theorem and its Corollary results in these double sums of $V(k,l)$ not being equal to $V(d,n)$. However, one has this Theorem.

Theorem 1.15.

$$\sum_{(k,l)\in X_{0011}(d,n)\cap\overline{V}} V(k,l) = V(d,n) - 1$$

Proof.

$$\begin{aligned} V(d,n) &= \sum_{(i,j)\in S_{00}(d,n)} V(i,j) \\ &= \sum_{k=0}^{n} V(d-1,k) \\ &= V(d-1,0) + \sum_{k=1}^{n} V(d-1,k) \\ &= 1 + \sum_{k=1}^{n} V(d-1,k) \\ &= 1 + \sum_{k=1}^{n}\sum_{j=0}^{d-1} V(j,k-1) \end{aligned}$$

by evaluating $V(d-1,k)$ over $S_{11}(d-1,k)$. The assertion follows now from the argument of Theorem 1.13. ☐

Corollary 1.16. *The Theorem is valid for summation over any of the following sets.*

$$\begin{aligned} X_{0011} &= X_{0101} \\ X_{1010} &= X_{1100} \\ X_{0010} &= X_{0100} \\ X_{1011} &= X_{1101} \end{aligned}$$

Proof. Direct from Theorem 1.13, its Corollary, and the Theorem ☐

One can obtain for the W function analogous results to Theorems 1.12, 1.13, and 1.15, and their Corollaries. Simply observe that

$$T_{pq}(d,n) = A[S_{pq}(d,n)] \quad \text{and} \quad Y_{pqrs}(d,n) = A[X_{pqrs}(d,n)],$$

where Y_{pqrs} is also given by the implied set equivalence in the expression

$$\sum_{(i,j)\in T_{pq}(d,n)}\ \sum_{(k,l)\in T_{rs}(i,j)} W(k,l) = \sum_{(k,l)\in Y_{pqrs}(d,n)} W(k,l)$$

Details are omitted.

6 Discrete differentiation on the cardinality functions

The V and W functions also allow various single and double discrete differentiations (differences.) First, for $F(d,n)$ an arbitrary function on J^2, let

$$\begin{aligned} F_0(d,n) &= F(d,n) - F(d-1,n) \\ F_1(d,n) &= F(d,n) - F(d,n-1), \end{aligned}$$

and for $i = 0, 1$, let

$$\begin{aligned} F_{i0}(d,n) &= F_i(d,n) - F_i(d-1,n) \\ F_{i1}(d,n) &= F_i(d,n) - F_i(d,n-1) \end{aligned}$$

Then,

Theorem 1.17. $F_{10}(d,n) = F_{01}(d,n)$

Proof. Direct from definitions □

For V, one has,

$$\begin{aligned} V_0(d,n) &= V(d,n-1) \\ V_1(d,n) &= V(d-1,n) \\ V_{00}(d,n) &= V(d,n-2) \\ V_{01}(d,n) &= V_{10}(d,n) = V(d-1,n-1) \\ V_{11}(d,n) &= V(d-2,n) \end{aligned}$$

For W, one has,

$$\begin{aligned} W_0(d,n) &= W(d,n) - W(d-1,n) \\ W_1(d,n) &= W(d-1,n-1) \\ W_{00}(d,n) &= W(d,n) - 2W(d-1,n) + W(d-2,n) \\ W_{01}(d,n) &= W_{10}(d,n) = W(d-1,n-1) - W(d-2,n-1) \\ W_{11}(d,n) &= W(d-2,n-2) \end{aligned}$$

Other difference equations may be specified. For example, let

$$\begin{aligned} F^0(d,n) &= \frac{F(d,n) - F(d-1,n)}{F(d-1,n)} = \frac{F_0(d,n)}{F(d-1,n)} \\ F^1(d,n) &= \frac{F(d,n) - F(d,n-1)}{F(d,n-1)} = \frac{F_1(d,n)}{F(d,n-1)} \end{aligned}$$

and for $i = 0, 1$, let

$$F^{i0}(d, n) = F^{i}(d, n) - F^{i}(d - 1, n)$$
$$F^{i1}(d, n) = F^{i}(d, n) - F^{i}(d, n - 1)$$

These formulas are intended to resemble $\log[F(d, n)]$ in continuous analysis. In general, $F^{10}(d, n) \neq F^{01}(d, n)$. However, one has this Theorem.

Theorem 1.18. *If* $F(d - 1, n) = F(d, n - 1)$ *then* $F^{10}(d, n) = F^{01}(d, n)$

Proof. Direct from definitions □

For V, one has,

$$\begin{aligned}
V^{0}(d, n) = V^{1}(n, d) &= \frac{n}{d} \\
V^{1}(d, n) = V^{0}(n, d) &= \frac{d}{n} \\
V^{00}(d, n) = V^{11}(n, d) &= -\frac{n}{d(d - 1)} \\
V^{01}(d, n) = V^{10}(n, d) &= \frac{1}{d} \\
V^{10}(d, n) = V^{01}(n, d) &= \frac{1}{n} \\
V^{11}(d, n) = V^{00}(n, d) &= -\frac{d}{n(n - 1)}
\end{aligned}$$

For W, one has,

$$\begin{aligned}
W^{0}(d, n) &= \frac{n + 1}{d} - 2 \\
W^{1}(d, n) &= \frac{d}{n - d} \\
W^{00}(d, n) &= -\frac{n + 1}{d(d - 1)} \\
W^{01}(d, n) &= \frac{1}{d} \\
W^{10}(d, n) &= \frac{n}{(n - d)(n - d + 1)} \\
W^{11}(d, n) &= -\frac{d}{(n - d)(n - d - 1)}
\end{aligned}$$

Note that

$$n = d \Longrightarrow V^{10} = V^{01},$$

and that

$$(n-d)(n-d-1) - nd = n^2 - 3nd + d^2 + n - d = 0 \Longrightarrow W^{10} = W^{01}$$

This latter polynomial is of substantial interest. Its investigation is the subject of the next Chapter.

Chapter 2

The equation of necessity for equal mixed partial derivatives

1 A set of solutions

By Theorem 1.18, $W^{10} = W^{01}$ if $\binom{n}{d-1} = \binom{n-1}{d}$, or equivalently, if

$$n^2 - 3nd + d^2 + n - d = 0 \tag{2.1}$$

To develop the integer solutions to this equation it is first necessary to establish a few Lemmas concerning the Fibonacci numbers, which play an important role. For a review see (Vorobyov 1963). The discourse then proceeds to exhibit a set of solutions, and to define relationships among these solutions in the form of nonlinear transformations of the plane.

Look first to the doubly infinite Fibonacci sequence,

$$(\dots, f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3, \dots) = (\dots, 2, -1, 1, 0, 1, 1, 2, \dots),$$

characterized by $f_0 = 0$, $f_1 = 1$, and $f_k = f_{k-2} + f_{k-1}$, for all integral k. For this sequence one has this Lemma.

Lemma 2.1. $f_k = (-1)^{k-1} f_{-k}$

Proof. By induction

The proposition is true for $k = 0, 1$. Then for $k \geqslant 2$,

$$\begin{aligned} f_k &= f_{k-2} + f_{k-1} \\ &= (-1)^{k-3} f_{-k+2} + (-1)^{k-2} f_{-k+1} \\ &= (-1)^{k-1} (f_{-k+2} - f_{-k+1}) \\ &= (-1)^{k-1} f_{-k} \end{aligned}$$

For $k < 0$,

$$f_{-k} = (-1)^{-k-1} f_k$$

Hence,

$$\begin{aligned} f_k &= (-1)^{k+1} f_{-k} \\ &= (-1)^{k-1} f_{-k} \end{aligned}$$ □

Lemma 2.2. $f_k = 0 \Longrightarrow k = 0$.

Proof. By induction on contrapositive
$f_1 = 1 > 0$, $f_2 = 1 > 0$. For $k \geqslant 3$, $f_k = f_{k-2} + f_{k-1} > 0$. For $k < 0$, $(-1)^{k-1} f_{-k} \neq 0$. □

Lemma 2.3. $f_k^2 = f_{k-1} f_{k+1} + (-1)^{k-1}$

Proof. By induction
The proposition is true for $k = 0$.

For $k \geqslant 1$,

$$\begin{aligned} f_k^2 &= f_k^2 + f_{k-1}^2 - f_{k-1}^2 + (-1)^{k-2} + (-1)^{k-1} \\ &= f_k^2 + f_{k-1}^2 - f_{k-2} f_k + (-1)^{k-1} \\ &= f_k (f_k - f_{k-2}) + f_{k-1}^2 + (-1)^{k-1} \\ &= f_k f_{k-2} + f_{k-1}^2 + (-1)^{k-1} \\ &= f_{k-1} (f_k + f_{k-1}) + (-1)^{k-1} \\ &= f_{k-1} f_{k+1} + (-1)^{k-1} \end{aligned}$$

For $k < 0$,

$$\begin{aligned} f_k^2 &= [(-1)^{k-1} f_{-k}]^2 = f_{-k}^2 \\ &= f_{-k-1} f_{-k+1} + (-1)^{-k-1} \\ &= (-1)^{-k-2} f_{k+1} (-1)^{-k} f_{k-1} + (-1)^{-k-1+2k} \\ &= f_{k-1} f_{k+1} + (-1)^{k-1} \end{aligned}$$ □

Now one may exhibit a set of solutions to Equation (2.1). Let

$$\begin{aligned} s_{2k} &= (d_k, n_k) \\ s_{2k+1} &= (d_{k+1}, n_k) \end{aligned}$$

where

$$\begin{aligned} d_k &= f_{2k-1} f_{2k} \\ n_k &= f_{2k} f_{2k+1} \end{aligned}$$

Then, one has this Theorem.

Theorem 2.4. *s_{2k} and s_{2k+1} are solutions to Equation (2.1).*

Proof.

Substituting s_{2k} into Equation (2.1),

$$\begin{aligned}
&(f_{2k}f_{2k+1})^2 + f_{2k}f_{2k+1} - 3f_{2k-1}f_{2k}f_{2k}f_{2k+1} + (f_{2k-1}f_{2k})^2 - f_{2k-1}f_{2k}\\
=&f_{2k}^2(f_{2k+1}^2 - 2f_{2k-1}f_{2k+1} + f_{2k-1}^2) - f_{2k}^2 f_{2k-1}f_{2k+1} + f_{2k}(f_{2k+1} - f_{2k-1})\\
=&f_{2k}^2(f_{2k+1} - f_{2k-1})^2 - f_{2k}^2 f_{2k-1}f_{2k+1} + f_{2k}^2\\
=&f_{2k}^2 f_{2k}^2 - f_{2k}^2(f_{2k-1}f_{2k+1} - 1)\\
=&f_{2k}^4 - f_{2k}^2 f_{2k}^2\\
=&f_{2k}^4 - f_{2k}^4\\
=&0
\end{aligned}$$

Substituting s_{2k+1} into Equation (2.1),

$$\begin{aligned}
&(f_{2k}f_{2k+1})^2 + f_{2k}f_{2k+1} - 3f_{2k+1}f_{2k+2}f_{2k}f_{2k+1} + (f_{2k+1}f_{2k+2})^2 - f_{2k+1}f_{2k+2}\\
=&f_{2k+1}^2(f_{2k}^2 - 2f_{2k}f_{2k+2} + f_{2k+2}^2) - f_{2k+1}^2 f_{2k}f_{2k+2} + f_{2k+1}(f_{2k+2} - f_{2k})\\
=&f_{2k+1}^2(f_{2k} - f_{2k+2})^2 - f_{2k+1}^2 f_{2k}f_{2k+2} - f_{2k+1}^2\\
=&f_{2k+1}^2 f_{2k+1}^2 - f_{2k+1}^2(f_{2k}f_{2k+2} + 1)\\
=&f_{2k+1}^4 - f_{2k+1}^2 f_{2k+1}^2\\
=&f_{2k+1}^4 - f_{2k+1}^4\\
=&0
\end{aligned}$$

□

Theorem 2.5. *The pairs (s_{2k}, s_{2k+1}) and (s_{2k}, s_{2k-1}) are conjugates in the quadratic reformulations of Equation (2.1).*

(2.2)
$$d^2 + (-1 - 3n)d + (n^2 + n) = 0$$

and

(2.3)
$$n^2 + (1 - 3d)n + (n^2 - d) = 0$$

Call the solutions to Equation (2.2), *n-conjugates*, and the solutions to Equation (2.3), *d-conjugates*.

Proof. It is necessary to show that $d_k + d_{k+1} = 3n_k + 1$, and that $n_k + n_{k-1} = 3d_k - 1$.

In the former case,

$$\begin{aligned}
d_k + d_{k+1} &= f_{2k-1}f_{2k} + f_{2k+1}f_{2k+2} \\
&= (f_{2k+1} - f_{2k})f_{2k} + f_{2k+1}(f_{2k} + f_{2k+1}) \\
&= f_{2k+1}f_{2k} - f_{2k}^2 + f_{2k+1}f_{2k} + f_{2k+1}^2 \\
&= 2f_{2k}f_{2k+1} + f_{2k+1}^2 - f_{2k}^2 \\
&= 2f_{2k}f_{2k+1} + (f_{2k}f_{2k+2} + 1) - f_{2k}^2 \\
&= 2f_{2k}f_{2k+1} + f_{2k}(f_{2k+2} - f_{2k}) + 1 \\
&= 2f_{2k}f_{2k+1} + f_{2k}f_{2k+1} + 1 \\
&= 3f_{2k}f_{2k+1} + 1 \\
&= 3n_k + 1
\end{aligned}$$

In the latter case,

$$\begin{aligned}
n_k + n_{k-1} &= f_{2k}f_{2k+1} + f_{2k-2}f_{2k-1} \\
&= (f_{2k-1} + f_{2k})f_{2k} + f_{2k-1}(f_{2k} - f_{2k-1}) \\
&= f_{2k}f_{2k-1} + f_{2k}^2 + f_{2k}f_{2k-1} - f_{2k-1}^2 \\
&= 2f_{2k-1}f_{2k} + f_{2k}^2 - f_{2k-1}^2 \\
&= 2f_{2k-1}f_{2k} + (f_{2k-1}f_{2k+1} - 1) - f_{2k-1}^2 \\
&= 2f_{2k-1}f_{2k} + f_{2k-1}(f_{2k+1} - f_{2k-1}) - 1 \\
&= 2f_{2k-1}f_{2k} + f_{2k-1}f_{2k} - 1 \\
&= 3f_{2k-1}f_{2k} - 1 \\
&= 3d_k - 1
\end{aligned}$$

□

One may remember solutions and conjugate relationships more easily by viewing the following diagram. Solutions are indicated by the joined arrows. Conjugate solutions are those with a common coordinate d_j or n_j.

$$\text{Conjugates of } \mathbf{n_k} \left\{ \begin{array}{ccc} f_{2k-1}f_{2k} = \mathbf{d_k} & \longrightarrow & n_{k-1} = f_{2k-2}f_{2k-1} \\ & \swarrow\!\!\nwarrow & \\ f_{2k+1}f_{2k+2} = d_{k+1} & \longleftarrow & \mathbf{n_k} = f_{2k}f_{2k+1} \end{array} \right\} \text{Conjugates of } \mathbf{d_k}$$

Diagram 1.

2 A transform relating the solutions

Next investigate a transformation on the real plane leaving the solution set $\{s_j\}$ invariant. As the transformation is nontrivial, it provides a means of recursive solution generation.

Theorem 2.6. *Let* T *be the nonlinear transformation in the plane defined by*

$$T(d,n) = \left[\frac{n(2n-d)}{n-d}, \frac{(2n-d)(3n-d)}{n-d}\right]$$

for $n \neq d$, $T(d,d) = (1,2)$. *Then*

$$\begin{aligned} T(s_{2k}) = T(d_k, n_k) \quad &= (d_{k+1}, n_{k+1}) = s_{2k+2} \\ T(s_{2k+1}) = T(d_{k+1}, n_k) &= (d_k, n_{k-1}) \quad = s_{2k-1}, \end{aligned}$$

i.e., T transforms the n-conjugates of s_{2k+1} and s_{2k} into the d-conjugates. Such a transformation (or its inverse) is called a *conjugate exchange*.

Proof.

In the former case,

$$\begin{aligned} n_k - d_k &= f_{2k}f_{2k+1} - f_{2k-1}f_{2k} \\ &= f_{2k}(f_{2k+1} - f_{2k-1}) \\ &= f_{2k}f_{2k} \\ &= f_{2k}^2 \\ 2n_k - d_k &= n_k + (n_k - d_k) \\ &= f_{2k}f_{2k+1} + f_{2k}^2 \\ &= f_{2k}(f_{2k+1} + f_{2k}) \\ &= f_{2k}f_{2k+2} \\ 3n_k - d_k &= n_k + (2n_k - d_k) \\ &= f_{2k}f_{2k+1} + f_{2k}f_{2k+2} \\ &= f_{2k}(f_{2k+1} + f_{2k+2}) \\ &= f_{2k}f_{2k+3} \end{aligned}$$

If $f_{2k} \neq 0$,

$$\begin{aligned} T(d_k, n_k) &= \left[\frac{(f_{2k}f_{2k+1})(f_{2k}f_{2k+2})}{f_{2k}^2}, \frac{(f_{2k}f_{2k+2})(f_{2k}f_{2k+3})}{f_{2k}^2}\right] \\ &= (f_{2k+1}f_{2k+2}, f_{2k+2}f_{2k+3}) \\ &= (d_{k+1}, n_{k+1}) \\ &= s_{2k+2} \end{aligned}$$

If $f_{2k} = 0$, then $k = 0$ by Lemma 2.2, and $T(d_0, n_0) = T(0, 0) = (1, 2) = (d_1, n_1) = s_2$.
In the latter case,

$$\begin{aligned}
n_k - d_{k+1} &= f_{2k}f_{2k+1} - f_{2k+1}f_{2k+2} \\
&= f_{2k+1}(f_{2k} - f_{2k+2}) \\
&= f_{2k+1}(-f_{2k+1}) \\
&= -f_{2k+1}^2 \\
2n_k - d_{k+1} &= n_k + (n_k - d_{k+1}) \\
&= f_{2k}f_{2k+1} - f_{2k+1}^2 \\
&= f_{2k+1}(f_{2k} - f_{2k+1}) \\
&= -f_{2k-1}f_{2k+1} \\
3n_k - d_{k+1} &= n_k + (2n_k - d_{k+1}) \\
&= f_{2k}f_{2k+1} - f_{2k-1}f_{2k+1} \\
&= f_{2k+1}(f_{2k} - f_{2k-1}) \\
&= f_{2k-2}f_{2k+1}
\end{aligned}$$

Lemma 2.2 implies $f_{2k+1} \neq 0$. Hence,

$$\begin{aligned}
T(d_{k+1}, n_k) &= \left[\frac{(f_{2k}f_{2k+1})(-f_{2k-1}f_{2k+1})}{-f_{2k+1}^2}, \frac{(-f_{2k-1}f_{2k+1})(f_{2k-2}f_{2k+1})}{-f_{2k+1}^2}\right] \\
&= (f_{2k-1}f_{2k}, f_{2k-2}f_{2k-1}) \\
&= (d_k, n_{k-1}) \\
&= s_{2k-1}
\end{aligned}$$

□

Theorem 2.7. *Let* T *be the nonlinear transformation in the plane defined by*

$$\widehat{T}(d, n) = \left[\frac{n(2d - n)(3d - n)}{d - n}, \frac{d(2d - n)}{d - n}\right]$$

for $n \neq d$, $\widehat{T}(d, d) = (-2, -1)$. *Then* $\widehat{T}$ *is the inverse of* T *on* $\{s_j\}$.

Proof. Directly verifiable □

Corollary 2.8. *If*

$$T(d, n) = [T_1(d, n), T_2(d, n)],$$

then

$$T^{-1}(d, n) = [T_2(n, d), T_1(n, d)]$$

3 Symmetries of the solution set

The solutions are symmetric around the line $n = -d$. More formally, one has this Theorem.

Theorem 2.9.

$$s_j = (d, n) \Longrightarrow s_{-j} = (-n, -d)$$

Proof.

Consider first the even subscripts.

$$\begin{aligned} s_{-2k} &= (d_{-k}, n_{-k}) \\ &= (f_{-2k-1} f_{-2k}, f_{-2k} f_{-2k+1}) \\ &= \left((-1)^{-2k-2}(-1)^{-2k-1} f_{2k+1} f_{2k}, (-1)^{-2k-1}(-1)^{-2k} f_{2k} f_{2k-1}\right) \end{aligned}$$

by Lemma 2.1.

$$\begin{aligned} &= (-f_{2k+1} f_{2k}, -f_{2k} f_{2k-1}) \\ &= (-n_k, -d_k) \end{aligned}$$

Next, consider the odd subscripts.

$$\begin{aligned} s_{-2k-1} &= (d_{-k}, n_{-k-1}) \\ &= (f_{-2k-1} f_{-2k}, f_{-2k-2} f_{-2k-1}) \\ &= \left((-1)^{-2k-2}(-1)^{-2k-1} f_{2k+1} f_{2k}, (-1)^{-2k-3}(-1)^{-2k-2} f_{2k+2} f_{2k+1}\right) \end{aligned}$$

by Lemma 2.1.

$$\begin{aligned} &= (-f_{2k+1} f_{2k}, -f_{2k+2} f_{2k+1}) \\ &= (-n_k, -d_{k+1}) \end{aligned}$$

□

Corollary 2.10.

$$s_j + s_{-j} = (n - d)(-1, 1),$$
$$s_j = s_{-j} = (n - d)(1, 1),$$

and

$$(s_j + s_{-j})(s_j - s_{-j}) = 0$$

A proof that the solutions $\{s_j\}$ are a complete set of solutions is given in the next Chapter.

Chapter 3

Recursive solutions on binary quadratics

1 The basic transforms

Examine the form

$$ay^2 + byx + cx^2 + dy + ex + f = 0 \tag{3.1}$$

This form in tensor notation is

$$A^{(2)}X^2 + A^{(1)}X + A^{(0)},$$

or in equivalent matrix notation is

$$X'QX + X'L + K = 0, \tag{3.2}$$

where

$$X = \begin{pmatrix} y \\ x \end{pmatrix}$$
$$Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$
$$L = \begin{pmatrix} d \\ e \end{pmatrix}$$
$$K = f$$

Define $\Delta = -4|Q| = b^2 - 4ac$. If $\Delta \geqslant 0$, the form is indefinite; otherwise the form is definite. Solution gives

$$y = \frac{-(bx + d) \pm p}{2a}, \tag{3.3}$$

where

$$p^2 = \Delta x^2 + 2(bd - 2ae)x + (d^2 - 4af), \tag{3.4}$$

and for which p must be integral for y to be. Equation (3.4) may be solved for x, giving

$$x = \frac{-(bd - 2ae) \pm q}{\Delta}, \tag{3.5}$$

where

$$q^2 = b^2d^2 + 4a^2e^2 - \Delta d^2 + 4\Delta af + \Delta p^2 \tag{3.6}$$

As with p, q must be integral for x to be. It follows from Equation (3.6), that

$$\Delta p^2 - q^2 = 4a(bde - ae^2 - cd^2 - \Delta f) \tag{3.7}$$

To reiterate, Equation (3.7) must have an integer solution for Equation (3.1) to have.

The Equations (3.3) and (3.5) define four affine transformations of the plane, given by

$$\begin{pmatrix} \pm p \\ \pm q \end{pmatrix} = \begin{pmatrix} b & 2a \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d \\ bd - 2ae \end{pmatrix}$$

Call these transformations A_{++}, A_{+-}, A_{-+}, and A_{--}, corresponding to the sign choices for p and q. For convenience, let $A = A_{++}$ and $\overline{A} = A_{+-}$. Then $A_{-+} = -\overline{A}$ and $A_{--} = -A$. These transformations have inverses given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-2a\Delta} \left[\begin{pmatrix} 0 & -2a \\ -\Delta & b \end{pmatrix} \begin{pmatrix} \pm p \\ \pm q \end{pmatrix} + 2a \begin{pmatrix} bd - 2ae \\ be - 2cd \end{pmatrix} \right],$$

where the sign choices are as above. Consider also the following linear transformations.

$$S \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{2ac} \begin{pmatrix} b^2 - 2ac & -b \\ -b\Delta & b^2 - 2ac \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

and its square root

$$\widehat{S} \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{2\sqrt{ac}} \begin{pmatrix} -b & 1 \\ \Delta & -b \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

These transformations have inverses given by

$$S^{-1} \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{2ac} \begin{pmatrix} b^2 - 2ac & b \\ b\Delta & b^2 - 2ac \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

and

$$\widehat{S}^{-1}\begin{pmatrix}p\\q\end{pmatrix} = \frac{1}{2\sqrt{ac}}\begin{pmatrix}-b & -1\\-\Delta & -b\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}$$

Note that S, $\widehat{S}$, S^{-1}, and $\widehat{S}^{-1}$ all have determinant $+1$, and are invariant under interchange of the variables x and y.

Finally, look at the affine transformation

$$T\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{ac}\begin{pmatrix}-ac & -ab\\bc & b^2-ac\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-ae\\be-cd\end{pmatrix} \tag{3.8}$$

and its inverse

$$T^{-1}\begin{pmatrix}x\\y\end{pmatrix} = \left[\frac{1}{ac}\begin{pmatrix}b^2-ac & ab\\-bc & -ac\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}bd-ae\\-cd\end{pmatrix}\right]$$

2 Theory relating the transforms

Under the assumption that $a \neq 0$, $c \neq 0$, $\Delta \neq 0$, in Equation (3.1), the following results derive readily (details omitted.)

Theorem 3.1.

$$\begin{pmatrix}x\\y\end{pmatrix} \textit{ solves Equation (3.1)} \iff A\begin{pmatrix}x\\y\end{pmatrix} \textit{ and } \overline{A}\begin{pmatrix}x\\y\end{pmatrix} \textit{ solve Equation (3.7).}$$

Theorem 3.2.

$$\begin{pmatrix}p\\q\end{pmatrix} \textit{ solves Equation (3.7)} \iff \widehat{S}\begin{pmatrix}p\\q\end{pmatrix} \textit{ and } S\begin{pmatrix}p\\q\end{pmatrix} \textit{ solve Equation (3.7).}$$

Theorem 3.3. $T = A^{-1}SA = \overline{A}^{-1}S\overline{A}$, *hence*

$$\begin{pmatrix}x\\y\end{pmatrix} \textit{ solves Equation (3.1)} \iff T\begin{pmatrix}x\\y\end{pmatrix} \textit{ solves Equation (3.1).}$$

Theorem 3.4 (Conjugate Exchange Property of T).

If $\begin{pmatrix}x_1\\y_1\end{pmatrix}$ *and* $\begin{pmatrix}x_2\\y_1\end{pmatrix}$ *are conjugate roots to Equation* (3.1),
then $T\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_2\\y_1\end{pmatrix}$, *such that* $\begin{pmatrix}x_2\\y_1\end{pmatrix}$ *and* $\begin{pmatrix}x_2\\y_2\end{pmatrix}$ *are conjugate.*

If $\begin{pmatrix}x_1\\y_1\end{pmatrix}$ *and* $\begin{pmatrix}x_1\\y_2\end{pmatrix}$ *are conjugate roots to Equation* (3.1),
then $T^{-1}\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_2\\y_2\end{pmatrix}$, *such that* $\begin{pmatrix}x_1\\y_2\end{pmatrix}$ *and* $\begin{pmatrix}x_2\\y_2\end{pmatrix}$ *are conjugate.*

Proof. Apply T, keeping in mind that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_1 \end{pmatrix}$ conjugate implies

$$x_1 + x_2 = \frac{-(by_1 + e)}{c}$$

Apply T^{-1}, keeping in mind that $\begin{pmatrix} x_2 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ conjugate implies

$$y_1 + y_2 = \frac{-(bx_2 + d)}{a}$$ □

Theorem 3.5. $\begin{pmatrix} x \\ y \end{pmatrix}$ *is a fixed point of* $\mathsf{T} \iff A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ *is a fixed point of* S.

Proof. If $\mathsf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $A\mathsf{T}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$. But $SA = AT$, hence $SA\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$. If $\mathsf{S}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$, then $A^{-1}\mathsf{S}\begin{pmatrix} p \\ q \end{pmatrix} = A^{-1}\begin{pmatrix} p \\ q \end{pmatrix}$. But $\mathsf{T}A^{-1} = A^{-1}\mathsf{S}$, hence $\mathsf{T}A^{-1}\begin{pmatrix} p \\ q \end{pmatrix} = A^{-1}\begin{pmatrix} p \\ q \end{pmatrix}$. □

Theorem 3.6. *The only fixed point of* S *is* $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Proof. $\mathsf{S}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ reduces to the pair of equations

$$\left.\begin{aligned} \Delta p - bq &= 0 \\ -b\Delta p + \Delta q &= 0 \end{aligned}\right\},$$

whence the conclusion readily derives. □

Theorem 3.7. *The only fixed point of* T *is* $\frac{1}{\Delta}\begin{pmatrix} 2ae - bd \\ 2cd - be \end{pmatrix}$.

Proof. Apply A^{-1} to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, using Theorems 3.5 and 3.6. □

Theorem 3.8.
The fixed point of T *is a solution to Equation* (3.1) $\iff bde - ae^2 - cd^2 - \Delta f = 0$.

Proof. Substitute $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ into Equation (3.7). □

Questions naturally arise concerning the functions mapping the algebraic varieties specified by Equation (3.1) onto their transforms A, S, and T. Assume now that the coefficients of Equation (3.1) are rational numbers. Note first that one may multiply Equation (3.1) by any nonzero rational number μ while preserving the variety. In vector notation, then, one may rewrite Equation (3.1) as

$$\mu(a,b,c,d,e,f)(y^2,yx,x^2,y,x,1)^{\mathsf{T}} = 0 \tag{3.9}$$

Let $\widetilde{A}$ be the function which maps expressions Equation (3.9), as points in rational 6-space, onto their A-transforms. There are five nonzero entries in that A-transform, hence one may view the form as a point in rational 5-space,

$$(b, 2a, d, \Delta, bd - 2ae) \tag{3.10}$$

If one allows the parameter μ to vary in Equation (3.9), Equation (3.10) becomes

$$\mu(b,2a,d,0,0) + \mu^2(0,0,0,\Delta, bd - 2ae) \tag{3.11}$$

Under suitable change of coordinates, Equation (3.11) becomes

$$\mu(1,0,0,0,0) + \mu^2(0,1,0,0,0)$$

Thus parametrized, one sees the curve to be a plane parabola embedded in the 5-space. $\widetilde{A}$ therefore maps lines Equation (3.9) into parabolas Equation (3.11). If one now identifies the parabolas with their tangent at the origin, $\widetilde{A}$ then maps points of projective 5-space onto points of projective 4-space.

The variety Equation (3.9) is in fact recoverable from the parabola Equation (3.11), save for displacements by the constant f. If one rewrite Equation (3.11) in terms of given constants,

$$\mu(a_{11}, a_{12}, b_1, 0, 0) + \mu^2(0,0,0,a_{21}, b_2), \tag{3.12}$$

then the variety is

$$\mu\left[(\tfrac{1}{2}a_{12})y^2 + (a_{11})yx + \left(\frac{a_{11}^2 - a_{21}}{2a_{12}}\right)x^2 + (b_1)y + \left(\frac{a_{11}b_1 - b_2}{a_{21}}\right)x\right] = 0 \tag{3.13}$$

Note, therefore, that $\widetilde{A}$ operating on the subspace for which $f = 0$ is invertible.

Look now to the map $\widetilde{S}$, which sends each variety Equation (3.2) onto its S-transform. If

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix},$$

then S is completely determined by its entries s_{11} and s_{12} (which prove.) $\widetilde{S}$ of a variety then is simply a point in a plane, as S is invariant on changes in μ. The forms are not uniquely recoverable, however, as only the product ac is determinable, not a and c individually. The specific determinations are

$$b = -\frac{s_{11}+1}{s_{12}}$$
$$ac = \frac{s_{11}+1}{2s_{12}^2}$$

The induced map $\widetilde{S}^*$ on the equivalence classes for which ac is a constant, is, however, invertible, up to displacements by d, e, and f.

Look finally to the map $\widetilde{T}$ which sends each variety Equation (3.9) onto its T-transform. As with S, T is independent of changes in μ. The T-transform is a point in 4-space, for if

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

knowledge of t_{12}, t_{21}, u_1, and u_2 is sufficient for a complete determination of the variety (which prove.)

$\widetilde{T}$ is invertible up to displacements by f. Specifically, choose b, then

$$c = -\frac{b}{t_{12}}$$
$$a = \frac{b}{t_{21}}$$
$$d = -u_2 \cdot a$$
$$e = \left(\frac{bd}{ac} - u_1\right) \cdot c$$

These values are rational, hence may be cleared to integers.

3 Equivalence classes under the unimodular group

If U be a unimodular linear transformation ($|U| = \pm 1$) with integer coefficients, then $X = U\widehat{X}$ defines a change of coordinates such that Equation (3.2) becomes

$$\widehat{X}'(U'QU)\widehat{X} + \widehat{X}'(U'L) + K = 0 \tag{3.14}$$

This form is a *unimodular transform* of Equation (3.2). Notre that $|U'QU| = |Q|$, hence the definiteness or indefiniteness of the form is invariant under the change. More importantly, the relation of unimodular transform is an equivalence relation, and the solution of any problem of a class implies solution of all others of the class. Specifically,

$$\begin{aligned} &\{\hat{s}_j\} \text{ and } \widehat{T}\begin{pmatrix}\hat{y}\\ \hat{x}\end{pmatrix}\\ \text{are}\quad &\left\{U^{-1}s_j\right\} \text{ and } U^{-1}TU\begin{pmatrix}\hat{y}\\ \hat{x}\end{pmatrix},\\ \text{given}\quad &U\begin{pmatrix}\hat{y}\\ \hat{x}\end{pmatrix} \text{ and } T\begin{pmatrix}y\\ x\end{pmatrix}\end{aligned}$$

4 Examples of the theory

Taking the example of the previous Chapter for elaboration (with $\binom{y}{x}$ replacing $\binom{n}{d}$),)

$$y^2 - 3yx + x^2 + y - x = 0, \tag{3.15}$$

one has the following transforms:

$$\begin{aligned} A\begin{pmatrix}x\\ y\end{pmatrix} &= \begin{pmatrix}-3 & 2\\ 5 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}1\\ -1\end{pmatrix}\\ \widehat{S}\begin{pmatrix}p\\ q\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}3 & 1\\ 5 & 3\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}\\ T\begin{pmatrix}x\\ y\end{pmatrix} &= \begin{pmatrix}-1 & 3\\ -3 & 8\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}1\\ 2\end{pmatrix}\end{aligned}$$

and

$$\Delta = 5$$

The substituted form of Equation (3.7) is $5p^2 - q^2 = 4$. Recalling that

$$\begin{aligned} s_{2k} &= (f_{2k-1}f_{2k}, f_{2k}f_{2k+1})\\ s_{2k+1} &= (f_{2k+1}f_{2k+2}, f_{2k}f_{2k+1}),\end{aligned}$$

the following relationships are valid.

$$\begin{aligned} T(s_{2k}) &= s_{2k+2}\\ T(s_{2k+1}) &= s_{2k-1}\\ A(s_{2k}) &= \begin{pmatrix}f_{4k-1}\\ f_{4k-2}+f_{4k}\end{pmatrix}\\ A(s_{2k-1}) &= \begin{pmatrix}-f_{4k-1}\\ f_{4k-2}+f_{4k}\end{pmatrix}\end{aligned}$$

The following diagrams may be helpful to visualize these relationships, first for even subscripts, then for odd subscripts.

$$\begin{array}{ccccccc}
s_{2k} & \longrightarrow & & \xrightarrow{T} & & \longrightarrow & s_{2k+2} & & \\
{\scriptstyle A}\downarrow & & & & & & {\scriptstyle A}\downarrow & & \\
A(s_{2k}) & \xrightarrow{\hat{S}} & \overline{A}(s_{-2k}) & \xrightarrow{\hat{S}} & A(s_{2k+2}) & \xrightarrow{\hat{S}} & \overline{A}(s_{-2k-2}) \\
 & & {\scriptstyle \overline{A}}\uparrow & & & & {\scriptstyle \overline{A}}\uparrow \\
 & & s_{-2k} & \longleftarrow & \xleftarrow{T} & \longleftarrow & s_{-2k-2}
\end{array}$$

Diagram 2.

$$\begin{array}{ccccccc}
s_{2k-1} & \longleftarrow & & \xleftarrow{T} & & \longleftarrow & s_{2k+1} \\
{\scriptstyle \overline{A}}\downarrow & & & & & & {\scriptstyle \overline{A}}\downarrow \\
\overline{A}(s_{2k-1}) & \xrightarrow{\hat{S}} & A(s_{-2k-1}) & \xrightarrow{\hat{S}} & \overline{A}(s_{2k+1}) & \xrightarrow{\hat{S}} & A(s_{-2k-3}) \\
 & & {\scriptstyle A}\uparrow & & & & {\scriptstyle A}\uparrow \\
 & & s_{-2k-1} & \longleftarrow & \xleftarrow{T} & \longleftarrow & s_{-2k-3}
\end{array}$$

Diagram 3.

For the reader interested in proving the above statements, these Lemmas should be useful.

Lemma 3.9.

$$f_k = \frac{\alpha^k - \beta^k}{\sqrt{5}},$$

where

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}$$

Be sure to prove for negative as well as non-negative k.

Lemma 3.10.

$$f_{2k-1}^2 + f_{2k}^2 = f_{4k+1}$$

Use Lemma 3.9.

Lemma 3.11.

$$5f_{2k-1}f_{2k} - 1 = f_{4k-2} + f_{4k}$$

Use Lemma 3.9.

One may also show that the nonlinear T of the previous Chapter is identical to T of this Chapter, when restricted to the lines of Equation (3.15), *i.e.*,

$$
\begin{aligned}
T\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \dfrac{y(2y-x)}{y-x} \\ \dfrac{(2y-x)(3y-x)}{y-x} \end{pmatrix} \\
&= \begin{pmatrix} -1 & 3 \\ -3 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\end{aligned}
$$

The completeness of solutions to Equation (3.15) generated by T and its powers appears now as a Theorem. A general completeness proof (or theory) to Equation (3.1) is elusive, as demonstrated by the discussion of examples to follow.

Theorem 3.12. *The powers of* T *applied to the conjugate solutions* $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *of Equation* (3.15) *provide a complete set of integer-valued solutions.*

Proof. Sketch, indirect

The fixed point of T is not a solution to Equation (3.15) because $bde - ae^2 - cd^2 = 1 \neq 0$ (Theorem 3.5.) Therefore T maps the variety Equation (3.15) monotonically into itself on both branches. Assume there is an integer-valued solution $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$ not yet considered on one branch. This solution lies between two adjacent solutions of the given sequence. Successive applications of T or T^{-1} will therefore transform these three points onto $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ on the one branch, or onto $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ on the other branch. Since $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$ must be integer valued, because T is closed on the integer plane, the contradiction obtains. □

Following are twelve examples, of which the above example is repeated first. The initial group of five are indefinite forms; the final group of seven are definite. The indefinite forms are of special interest owing to their frequency of occurrence.[1]

Note that $\Delta < 0$ implies that Equation (3.7) defines an ellipse, and hence that Equation (3.1) defines and ellipse, since A^{-1} is a homeomorphism. This fact follows because

$$4a(bde - ae^2 - cd^2) < 4abde - 4a^2e^2 - b^2d^2 = -(2ae - bd)^2 < 0$$

[1] Two measures of frequency come to mind. (a) Let a, b, c be drawn independently from a (continuous) uniform distribution on [r,r]. The probability of definiteness is $2\int_0^r \int_0^r \int_{-2\sqrt{ac}}^{2\sqrt{ac}} db\, da\, dc = 4/9$. (b) If one restricts attention to similarity classes of diagonal matrices, then in only two cases (all positive or all negative eigenvalues) of 2^d possible combinations, d the dimension, is the resulting form definite.

Otherwise Equation (3.1) is an hyperbola. For all examples, $f = 0$. Therefore, $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}0\\-d/a\end{pmatrix}$, and $\begin{pmatrix}-e/c\\0\end{pmatrix}$ are always solutions. Observing these ideas and the specification of T, Equation (3.8) implies that all powers of T on $\begin{pmatrix}0\\0\end{pmatrix}$ and its conjugates are integer valued if a divides b and d, and if c divides b and e. These requirements are not met by Examples 4 through 8, but are by the others. $\widehat{S}$ is the choice for illustration if rational, *i.e.*, if $\sqrt{ac}$ is rational; otherwise S is the choice.

Completeness proofs as for Example 1 (Theorem 3.12) also exist for Examples 2 and 3. Example 9 (a circle) exhibits the "Pythagorean" points at the ends of hypotenuse-radii of 3-4-5 triangles.

Example 5 is the intersection of two straight lines. This condition is equivalent to the fixed point's being a solution, for if $bde - ae^2 - cd^2 = 0$, then multiplying Equation (3.1) by de, and substituting for the coefficient of yx, allows the factorization

$$(aey + cdx + de)(dy + ex) = 0$$

One easily checks the intersection of the two lines to be the fixed point.

The most difficult outstanding problem is that of classifying forms of the type Equation (3.1) by methods for recursive generation of complete solution sets. The classic reference for the indefinite case is that of Rignaux (See Equation (P-4).) The given formula was promised to generate new integer solutions. The method fails, however, on Example 4, where T bypasses two integer points on each branch. Using these "interior" solutions in the Rignaux formula gives inconsistent determinations for μ.

The study of indefinite forms would take a major stride if this conjecture were proven (or disproved.)

Conjecture 3.13. *If* T *and its powers generate, for indefinite forms with* $\begin{pmatrix}0\\0\end{pmatrix}$ *a solution, integer sets on any of the following points or their conjugates, then the union of such sets is a complete set of solutions.*

$$\begin{pmatrix}0\\0\end{pmatrix}$$

$$A^{-1}\widehat{S}A\begin{pmatrix}0\\0\end{pmatrix}$$

$$A^{-1}\widehat{S}\overline{A}\begin{pmatrix}0\\0\end{pmatrix}$$

$$\overline{A}^{-1}\widehat{S}A\begin{pmatrix}0\\0\end{pmatrix}$$

$$\overline{A}^{-1}\widehat{S}\overline{A}\begin{pmatrix}0\\0\end{pmatrix}$$

All indefinite Examples given are in agreement with the conjecture.

A graph appears with each example showing a few central points and the effect of the transformation T.

Example 3.1.

$$y^2 - 3yx + x^2 + y - x = 0$$

$$\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -3 & 2 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\widehat{S}\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\
T\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 & 3 \\ -3 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= 5 \\
5p^2 - q^2 &= 4
\end{aligned}$$

$$\begin{array}{ccccccccccccc}
\xrightarrow{T} & \longrightarrow & \binom{-2}{-1} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{0}{0} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{1}{2} & \longrightarrow & \xrightarrow{T} \quad \longrightarrow \\
& & {\scriptstyle A}\downarrow & & & & {\scriptstyle A}\downarrow & & & & {\scriptstyle A}\downarrow & & \\
& \xrightarrow{\hat{S}} & \binom{5}{-11} & \xrightarrow{\hat{S}} & \binom{2}{-4} & \xrightarrow{\hat{S}} & \binom{1}{-1} & \xrightarrow{\hat{S}} & \binom{1}{1} & \xrightarrow{\hat{S}} & \binom{2}{4} & \xrightarrow{\hat{S}} & \binom{5}{11} \quad \xrightarrow{\hat{S}} \\
& & & & {\scriptstyle \overline{A}}\uparrow & & & & {\scriptstyle \overline{A}}\uparrow & & & & {\scriptstyle \overline{A}}\uparrow \\
& \longleftarrow & \xleftarrow{T} & \longleftarrow & \binom{1}{2} & \longleftarrow & \xleftarrow{T} & \longleftarrow & \binom{0}{0} & \longleftarrow & \xleftarrow{T} & \longleftarrow & \binom{-2}{-1} \quad \longleftarrow \quad \xleftarrow{T}
\end{array}$$

$$\begin{array}{ccccccccccccc}
& \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{1}{0} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{0}{-1} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{-2}{-6} \quad \longrightarrow \quad \xrightarrow{T} \\
& & & & {\scriptstyle A}\downarrow & & & & {\scriptstyle A}\downarrow & & & & {\scriptstyle A}\downarrow \\
& \xrightarrow{\hat{S}} & \binom{-5}{11} & \xrightarrow{\hat{S}} & \binom{-2}{4} & \xrightarrow{\hat{S}} & \binom{-1}{1} & \xrightarrow{\hat{S}} & \binom{-1}{-1} & \xrightarrow{\hat{S}} & \binom{-2}{-4} & \xrightarrow{\hat{S}} & \binom{-5}{-11} \quad \xrightarrow{\hat{S}} \\
& & {\scriptstyle \overline{A}}\uparrow & & & & {\scriptstyle \overline{A}}\uparrow & & & & {\scriptstyle \overline{A}}\uparrow & & \\
\xleftarrow{T} & \longleftarrow & \binom{-2}{-6} & \longleftarrow & \xleftarrow{T} & \longleftarrow & \binom{0}{-1} & \longleftarrow & \xleftarrow{T} & \longleftarrow & \binom{1}{0} & \longleftarrow & \xleftarrow{T} \quad \longleftarrow
\end{array}$$

Example 3.2.

$$2y^2 + 6yx + 3x^2 + 4y + 9x = 0$$

$$\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 6 & 4 \\ 12 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -12 \end{pmatrix} \\
S\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 4 & -1 \\ -12 & 4 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\
T\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 & -2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 7 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= 12 \\
12p^2 - q^2 &= 48
\end{aligned}$$

$$\begin{array}{ccccccc}
\xrightarrow{T} & \binom{1}{-2} & \xrightarrow{T} & \binom{0}{0} & \xrightarrow{T} & \binom{-3}{7} & \xrightarrow{T} \\
 & A\big\downarrow & & A\big\downarrow & & A\big\downarrow & \\
\xrightarrow[S]{} & \binom{2}{0} & \xrightarrow[S]{} & \binom{4}{-12} & \xrightarrow[S]{} & \binom{14}{-48} & \xrightarrow[S]{}
\end{array}$$

$$\begin{array}{ccccccc}
\xrightarrow{T} & \binom{-3}{0} & \xrightarrow{T} & \binom{0}{-2} & \xrightarrow{T} & \binom{1}{-3} & \xrightarrow{T} \\
 & A\big\downarrow & & A\big\downarrow & & A\big\downarrow & \\
\xrightarrow[S]{} & \binom{-14}{-48} & \xrightarrow[S]{} & \binom{-4}{-12} & \xrightarrow[S]{} & \binom{-2}{0} & \xrightarrow[S]{}
\end{array}$$

Example 3.3.

$$y^2 - 3yx + x^2 + 3y - 2x = 0$$

$$\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -3 & 2 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix} \\
\widehat{S}\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\
T\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 & 3 \\ -3 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= 5 \\
5p^2 - q^2 &= 20
\end{aligned}$$

$$\begin{array}{ccccccccccccc}
\binom{-7}{-3} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{0}{0} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{2}{3} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{9}{21} \\
A\big\downarrow & & & & A\big\downarrow & & & & A\big\downarrow & & & & A\big\downarrow \\
\binom{18}{-40} & \xrightarrow{\widehat{S}} & \binom{7}{-15} & \xrightarrow{\widehat{S}} & \binom{3}{-5} & \xrightarrow{\widehat{S}} & \binom{2}{0} & \xrightarrow{\widehat{S}} & \binom{3}{5} & \xrightarrow{\widehat{S}} & \binom{7}{15} & \xrightarrow{\widehat{S}} & \binom{18}{40} \\
 & & \overline{A}\big\uparrow & & & & \overline{A}\big\uparrow & & & & \overline{A}\big\uparrow & & \\
 & & \binom{4}{8} & \longleftarrow & \xleftarrow{T} & \longleftarrow & \binom{1}{1} & \longleftarrow & \xleftarrow{T} & \longleftarrow & \binom{-2}{-1} & &
\end{array}$$

$$\begin{array}{ccccccccccccc}
\binom{9}{3} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{2}{0} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{0}{-3} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{-7}{-21} \\
A\big\downarrow & & & & A\big\downarrow & & & & A\big\downarrow & & & & A\big\downarrow \\
\binom{-18}{40} & \xrightarrow{\widehat{S}} & \binom{-7}{15} & \xrightarrow{\widehat{S}} & \binom{-3}{5} & \xrightarrow{\widehat{S}} & \binom{-2}{0} & \xrightarrow{\widehat{S}} & \binom{-3}{-5} & \xrightarrow{\widehat{S}} & \binom{-7}{-15} & \xrightarrow{\widehat{S}} & \binom{-18}{-40} \\
 & & \overline{A}\big\uparrow & & & & \overline{A}\big\uparrow & & & & \overline{A}\big\uparrow & & \\
 & & \binom{-2}{-8} & \longleftarrow & \xleftarrow{T} & \longleftarrow & \binom{1}{-1} & \longleftarrow & \xleftarrow{T} & \longleftarrow & \binom{4}{1} & &
\end{array}$$

Example 3.4.

$$2y^2 - 8yx + 3x^2 + 12y - 9x = 0$$

$$\begin{aligned} A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -8 & 4 \\ 40 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 12 \\ -60 \end{pmatrix} \\ S\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} 13 & 2 \\ 80 & 13 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\ T\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3}\left[\begin{pmatrix} -3 & 8 \\ -12 & 29 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 9 \\ 18 \end{pmatrix}\right] \end{aligned}$$

and

$$\begin{aligned} \Delta &= 40 \\ 40p^2 - q^2 &= 2160 \end{aligned}$$

$$\begin{array}{ccccccccc} \binom{0}{0} & & \binom{1}{1} & & \binom{2}{3} & & \binom{3}{6} & \xleftarrow{T} & \binom{0}{0} \; \cdots \\ A\downarrow & & A\downarrow & & A\downarrow & & A\downarrow & & A\downarrow \\ \binom{12}{-60} & & \binom{8}{-20} & & \binom{8}{20} & & \binom{12}{60} & \xleftarrow{S} & \binom{12}{-60} \; \cdots \end{array}$$

$$\begin{array}{ccccccccc} \binom{3}{0} & & \binom{2}{-1} & & \binom{1}{-3} & & \binom{0}{-6} & \xleftarrow{T} & \binom{3}{0} \; \cdots \\ A\downarrow & & A\downarrow & & A\downarrow & & A\downarrow & & A\downarrow \\ \binom{-12}{60} & & \binom{-8}{20} & & \binom{-8}{-20} & & \binom{-12}{-60} & \xleftarrow{S} & \binom{-12}{60} \; \cdots \end{array}$$

Example 3.5.

$$
\begin{gathered}
-6y^2 + 11yx + 10x^2 + 16y - 40x = 0 \\
(2x + 3y - 8)(5x - 2y) = 0
\end{gathered}
$$

$$
\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 11 & -12 \\ 361 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 16 \\ -304 \end{pmatrix} \\
S\begin{pmatrix} p \\ q \end{pmatrix} &= -\frac{1}{120}\begin{pmatrix} 241 & -11 \\ 3971 & 240 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\
T\begin{pmatrix} x \\ y \end{pmatrix} &= -\frac{1}{60}\left[\begin{pmatrix} 60 & 66 \\ 110 & 181 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -240 \\ -600 \end{pmatrix}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta &= 361 \\
361p^2 - q^2 &= 0
\end{aligned}
$$

$$
\begin{array}{cccccc}
\binom{0}{0}^{*} & \binom{1}{2}^{\dagger} & \binom{2}{5}^{*} & \binom{4}{10}^{*} & \xleftarrow{T} & \binom{0}{0}^{*} \cdots \\
{\scriptstyle A}\downarrow & {\scriptstyle A}\downarrow & {\scriptstyle A}\downarrow & {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow \\
\binom{16}{-304} & \binom{3}{57} & \binom{-22}{418} & \binom{-60}{1140} & \xleftarrow{S} & \binom{16}{-304} \cdots
\end{array}
$$

---

* $(5x - 2y) = 0$ contains these points.

† $(2x + 3y - 8) = 0$ contains this point.

Note: $\alpha x + \beta y - (\alpha + 2\beta) = 0$ for any (α, β) would contain $\binom{1}{2}$.

Example 3.6.

$$y^2 + yx + 2x^2 + y - 2x = 0$$

$$\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ -7 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\
S\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{4}\begin{pmatrix} -3 & -1 \\ 7 & -3 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\
T\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{2}\left[\begin{pmatrix} -2 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}\right]
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= -7 \\
7p^2 + q^2 &= 32
\end{aligned}$$

$$\begin{array}{ccccccc}
\binom{0}{0} & \xrightarrow{T} & \binom{1}{-2} & \xrightarrow{T} & \binom{1}{0} & \xrightarrow{T} & \binom{0}{-1} \\
{\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow \\
\binom{1}{5} & \xrightarrow{S} & \binom{-2}{-2} & \xrightarrow{S} & \binom{2}{-2} & \xrightarrow{S} & \binom{-1}{5}
\end{array}$$

Example 3.7.

$$2y^2 + yx + x^2 - 2y + x = 0$$

This is an interchange of y and x of Example 6.

$$\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 4 \\ -7 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ -6 \end{pmatrix} \\
S\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{4}\begin{pmatrix} -3 & -1 \\ 7 & -3 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\
T\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{2}\left[\begin{pmatrix} -2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}\right]
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= -7 \\
7p^2 + q^2 &= 64
\end{aligned}$$

$$\begin{array}{ccccccccc}
\binom{-1}{0} & \xrightarrow{T} & \binom{0}{1} & \xrightarrow{T} & \binom{-2}{1} & \xrightarrow{T} & \binom{0}{0} & & \\
A\downarrow & & A\downarrow & & A\downarrow & & A\downarrow & & \\
\binom{-3}{1} & \xrightarrow{S} & \binom{2}{-6} & \xrightarrow{S} & \binom{0}{8} & \xrightarrow{S} & \binom{-2}{-6} & \xrightarrow{S} & \binom{3}{1}
\end{array}$$

Example 3.8.

$$4y^2 + yx + x^2 - 4y + x = 0$$

$$\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 8 \\ -15 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ -12 \end{pmatrix} \\
\widehat{S}\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{4}\begin{pmatrix} -1 & 1 \\ -15 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\
T\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{4}\left[\begin{pmatrix} -4 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ 5 \end{pmatrix}\right]
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= -15 \\
15p^2 + q^2 &= 384
\end{aligned}$$

$$\begin{array}{ccccccccccc}
 & & \binom{-2}{1} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{0}{0} & & & & \\
 & & {\scriptstyle A}\downarrow & & & & {\scriptstyle A}\downarrow & & & & \\
\binom{-5}{3} & \xrightarrow{\widehat{S}} & \binom{2}{18} & \xrightarrow{\widehat{S}} & \binom{4}{-12} & \xrightarrow{\widehat{S}} & \binom{-4}{-12} & \xrightarrow{\widehat{S}} & \binom{-2}{18} & \xrightarrow{\widehat{S}} & \binom{5}{3} \\
 & & & & {\scriptstyle \overline{A}}\uparrow & & & & {\scriptstyle \overline{A}}\uparrow & & \\
 & & & & \binom{0}{1} & \longrightarrow & \xleftarrow{T} & \longrightarrow & \binom{-1}{0} & &
\end{array}$$

Example 3.9.

$$
\begin{gathered}
y^2 + x^2 - 10x = 0 \\
\left(y^2 + (x-5)^2 = 5^2\right)
\end{gathered}
$$

$$
\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 & 2 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 20 \end{pmatrix} \\
\widehat{S}\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\
T\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta &= -4 \\
4p^2 + q^2 &= 400
\end{aligned}
$$

$$
\begin{array}{ccccccccccccc}
\binom{0}{0} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{10}{0} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{0}{0} & & & & \cdots \\
A\big\downarrow & & & & A\big\downarrow & & & & A\big\downarrow & & & & \\
\binom{0}{20} & \xrightarrow{\widehat{S}} & \binom{10}{0} & \xrightarrow{\widehat{S}} & \binom{0}{-20} & \xrightarrow{\widehat{S}} & \binom{-10}{0} & \xrightarrow{\widehat{S}} & \binom{0}{20} & & & & \cdots \\
 & & A\big\uparrow & & & & A\big\uparrow & & & & & & \\
 & & \binom{5}{5} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{5}{-5} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{5}{5} & & \cdots
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
\binom{1}{3} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{9}{-3} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{1}{3} & & & & \cdots \\
A\big\downarrow & & & & A\big\downarrow & & & & A\big\downarrow & & & & \\
\binom{6}{16} & \xrightarrow{\widehat{S}} & \binom{8}{-12} & \xrightarrow{\widehat{S}} & \binom{-6}{-16} & \xrightarrow{\widehat{S}} & \binom{-8}{12} & \xrightarrow{\widehat{S}} & \binom{6}{16} & & & & \cdots \\
 & & A\big\uparrow & & & & A\big\uparrow & & & & & & \\
 & & \binom{8}{4} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{2}{-4} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{8}{4} & & \cdots
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
\binom{2}{4} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{8}{-4} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{2}{4} & & & & \cdots \\
A\big\downarrow & & & & A\big\downarrow & & & & A\big\downarrow & & & & \\
\binom{8}{12} & \xrightarrow{\widehat{S}} & \binom{6}{-16} & \xrightarrow{\widehat{S}} & \binom{-8}{-12} & \xrightarrow{\widehat{S}} & \binom{6}{-16} & \xrightarrow{\widehat{S}} & \binom{8}{12} & & & & \cdots \\
 & & A\big\uparrow & & & & A\big\uparrow & & & & & & \\
 & & \binom{9}{3} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{1}{-3} & \longrightarrow & \xrightarrow{T} & \longrightarrow & \binom{9}{3} & & \cdots
\end{array}
$$

Example 3.10.

$$y^2 - yx + x^2 + y - x = 0$$

$$\begin{aligned} A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 & 2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ S\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\ T\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \Delta &= -3 \\ 3p^2 + q^2 &= 4 \end{aligned}$$

$$\begin{array}{ccccccccc} \binom{0}{0} & \xrightarrow{T} & \binom{1}{0} & \xrightarrow{T} & \binom{0}{-1} & \xrightarrow{T} & \binom{0}{0} & & \cdots \\ {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow & & \\ \binom{1}{1} & \xrightarrow{S} & \binom{0}{-2} & \xrightarrow{S} & \binom{-1}{1} & \xrightarrow{S} & \binom{1}{1} & & \cdots \end{array}$$

Example 3.11.

$$y^2 + x^2 + y - x = 0$$

$$\begin{aligned} A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 & 2 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ S\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\ T\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \Delta &= -4 \\ 4p^2 + q^2 &= 8 \end{aligned}$$

$$\begin{array}{ccccccc} \binom{0}{0} & \xrightarrow{T} & \binom{1}{-1} & \xrightarrow{T} & \binom{0}{0} & & \cdots \\ A\downarrow & & A\downarrow & & A\downarrow & & \\ \binom{1}{2} & \xrightarrow{S} & \binom{-1}{-2} & \xrightarrow{S} & \binom{1}{2} & & \cdots \end{array}$$

$$\begin{array}{ccccccc} \binom{0}{-1} & \xrightarrow{T} & \binom{1}{0} & \xrightarrow{T} & \binom{0}{-1} & & \cdots \\ A\downarrow & & A\downarrow & & A\downarrow & & \\ \binom{-1}{2} & \xrightarrow{S} & \binom{1}{-2} & \xrightarrow{S} & \binom{-1}{2} & & \cdots \end{array}$$

Example 3.12.

$$y^2 + yx + x^2 + y - x = 0$$

$$\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
S\begin{pmatrix} p \\ q \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\
T\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= -3 \\
3p^2 + q^2 &= 12
\end{aligned}$$

$$\begin{array}{ccccccccc}
\binom{0}{0} & \xrightarrow{T} & \binom{1}{-2} & \xrightarrow{T} & \binom{2}{-1} & \xrightarrow{T} & \binom{0}{0} & & \cdots \\
A\big\downarrow & & A\big\downarrow & & A\big\downarrow & & A\big\downarrow & & \\
\binom{1}{3} & \xrightarrow{S} & \binom{-2}{0} & \xrightarrow{S} & \binom{1}{-3} & \xrightarrow{S} & \binom{1}{3} & & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\binom{2}{-2} & \xrightarrow{T} & \binom{1}{0} & \xrightarrow{T} & \binom{0}{-1} & \xrightarrow{T} & \binom{2}{-2} & & \cdots \\
A\big\downarrow & & A\big\downarrow & & A\big\downarrow & & A\big\downarrow & & \\
\binom{-1}{-3} & \xrightarrow{S} & \binom{2}{0} & \xrightarrow{S} & \binom{-1}{3} & \xrightarrow{S} & \binom{-1}{-3} & & \cdots
\end{array}$$

Chapter 4

Recursive solutions on n-ary quadratics

1 An algorithm

Consider the general quadratic

$$X'QX + X'L + K = 0 \tag{4.1}$$

where Q is an $n \times n$ symmetric matrix, L is an n-vector, and K is a scalar. Impose the restriction that Equation (4.1) be quadratic in any pair of variables. For convenience, let $K = 0$, specifically to avoid two problems —

(a) the problem of no locus, *e.g.*, $x_1^2 + x_2^2 + 1^2 = 0$, and

(b) the problem of no axial intersection, *e.g.*, $(x_1 - 2)^2 + (x_2 - 2)^2 - 1^2 = 0$

The requirement $K = 0$ avoids both difficulties, since the zero vector is always a solution.

The plan of search, say in a mathematical programming problem, is to begin somewhere, *e.g.*, the origin, and to choose a *plane* of search, perhaps by some gradient criterion. Then compute the conjugate exchange affine transformation T for that plane, and search. (T exists, because Equation (4.1) is quadratic in the chosen pair of variables.) When a new direction is desired, one uses the coordinates of the latest trial point to compute the new T, *etc*.

Thus far, nothing has been assumed about the nature of the problem for which one searches the manifold. Special considerations, such as convexity, may influence the search.[1] Also, one may wish to use search methods borrowed from continuous analysis,

[1]It is well to note that Q of Equation (4.1) may not be definite, guaranteeing convexity (or concavity.) Indefinite forms, however, can bound unions of convex regions, *e.g.*, $y^2 - x^2 + 2x < 0$. Also, note the subtle difference between a convex function and a convex region. If $f(x)$ be the form Equation (4.1), then this theorem refers to a function $f(\alpha X_1 + (1-\alpha)X_2) < \alpha f(X_1) + (1-\alpha)f(X_2)$ for all $X_1 \neq X_2 \iff (X_1 - X_2)'Q(X_1 - X_2) > 0$, *i.e.*, $\iff$ Q is positive definite. This corollary refers to a region: If Q is positive definite, then $f(X_1) < k$, $f(X_2) < k$ imply $f(\alpha X_1 + (1-\alpha)X_2) < \alpha f(X_1) + (1-\alpha)f(X_2) < k$, *i.e.*, $\{X \mid f(X) < k\}$ is convex.

to help localize a probable optimum. Such tools include gradient, conjugate gradient, and penalty function techniques.

The search normally terminates when no further move in the constraint space improves the objective. In this regard, finiteness of the search is an important consideration and may depend on convexity attributes, boundedness of the constraint space, or other criteria such as choice of starting point.

2 Implications for real searches

One may also use techniques advanced herein when the requirement of integrality need not be met. Two additional procedures come to mind. One concerns the controlled refinement of conjugate exchange search. The other concerns binary search, once an optimum is isolated.

In the former case, note that by transformation under the affine group, any nontrivial binary quadratic is equivalent to one of the following three curves.

(a) an hyperbola, $y^2 - x^2 = 12$

(b) a circle, $y^2 + x^2 = 12$

(c) two lines, $y^2 - x^2 = 0$

These classes are determined by $\Delta > 0$, $\Delta < 0$, and $bde - ae^2 - cd^2 = 0$, respectively, with the last taking precedence. One may search these three curves to any desired refinement by applying linear transformations to solutions. One has this Theorem.

Theorem 4.1.

For the hyperbola,

$$U_1 = \begin{pmatrix} \sqrt{1+p^2} & p \\ p & \sqrt{1+p^2} \end{pmatrix}$$

for the circle,

$$U_2 = \begin{pmatrix} \sqrt{1-p^2} & p \\ -p & \sqrt{1-p^2} \end{pmatrix}$$

for the lines,

$$U_3 = \frac{1}{2} \begin{pmatrix} p + 1/p & p - 1/p \\ p - 1/p & p + 1/p \end{pmatrix}$$

map solutions to solutions on the respective curves, for any p *leaving* U_j *real.*

Proof. Simple substitutions effect the desired conclusion. □

As yet, the $\{U_j\}$ are not conjugate exchange searches. If, however, one changes coordinates suitably, transformations similar to the $\{U_j\}$ *are* conjugate exchanges.

Theorem 4.2.

For the hyperbola, the shear

$$H_1 = \begin{pmatrix} 1 & 0 \\ 1 + \sqrt{1+p^2} & 1 \end{pmatrix}$$

for the circle, the shear

$$H_2 = \begin{pmatrix} 1 & 0 \\ \dfrac{1 + \sqrt{1-p^2}}{p} & 1 \end{pmatrix}$$

for the lines, the shear

$$H_3 = \begin{pmatrix} 1 & 0 \\ \dfrac{p+1}{p-1} & 1 \end{pmatrix}$$

are such that $T_j = H_j^{-1} U_j^{-1} H_j$ *operating on the curves transformed by*

$$\begin{pmatrix} x \\ y \end{pmatrix} = H_j \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

are conjugate exchange.

Proof. Direct computation establishes the result. □

For binary search, it is sufficient to note that the following expressions are parametric representations of the curves.

For the hyperbola,

$$x = \tan\theta$$
$$y = \sec\theta$$

for the circle,

$$x = \cos\theta$$
$$y = \sin\theta$$

for the lines,

$$x = \tan\theta$$
$$y = \pm\tan\theta$$

A search, then, on θ provides a search on the curve.

Chapter 5

Directions for additional thought and experiment

(a) Proof or disproof of the completeness conjecture of Chapter 3

(b) Direct searches of more than two dimensions on hyperquadratic surfaces, or on varieties defined by more than one polynomial

For multidimensional search, one could look to affine substitutions into the form

$$X'QX + X'L + K = 0 \tag{5.1}$$

Replacing X by $VX + W$, Equation 5.1 becomes

$$X'(V'QV)X + X'(V'Q)W + W'(QV)X + \underline{W'(Q)W} + X'(V')L + \underline{W'L} + \underline{K} = 0$$

If $K = 0$, $X = 0$ is a solution, so $V(0) + W = W$ is a solution, eliminating the underlined terms. The challenge, therefore, reduces to finding V and W for which

$$X'(V'QV)X + X'(V'Q)W + W'(QV)X + X'(V')L = 0,$$

if Equation 5.1 holds.

Even if one finds satisfactory V and W, however, higher dimensional direct search has the disadvantage of not being a conjugate exchange. This conclusion derives from the fact that every point on the surface has three or more conjugates, and exchanging in two dimensions leaves other dimensions unaffected.

(c) An algebraic investigation of polynomial function rings from the viewpoint of simplex functions

(d) Analysis of the limiting distribution of the simplex overflow distribution

(e) Analysis of time and space considerations for use of linearized computer indexing of higher degree polynomials

(f) Specification and test of computer algorithms for searching and solving nonlinear programs in integers

(g) Reduction of the study of heterogeneous forms to the study of homogeneous forms

In this regard note that any form

$$A^{(d)}x^d + A^{(d-1)}x^{d-1} + \cdots + A^{(0)} = 0 \qquad (5.2)$$

in $n+1$ variables $x = (x_0, x_1, \ldots, x_n)$ may be "homogenized" to

$$B^{(d)}\bar{x}^d = 0 \qquad (5.3)$$

in $n+2$ variables $\bar{x} = (x_0, x_1, \ldots, x_n, x_{n+1})$ by setting

$$b_{i_0 i_1 \cdots i_{k-1} j_k \cdots j_{d-1}} = a_{i_0 i_1 \cdots i_{k-1}}$$

for $j_m = n+1, k \leqslant m \leqslant d-1$. Then $x^0 = (x^0_0, x^0_1, \ldots, x^0_n)$ is a solution to Equation 5.2 if and only if $\bar{x}^0 = (x^0_0, x^0_1, \ldots, x^0_n, 1)$ is a solution to Equation 5.3. This homogenized function, written $B^{(d)} = \text{homogen}(A^{(d)}, A^{(d-1)}, \ldots, A^{(0)})$ is one-to-one and onto (bijective,) and is a monoid isomorphism, with polynomial multiplication taken as the compositor. (Observe the consistency here with the result of Theorem 1.7, which specifies that $V(d, n+1) = \sum_{k=0}^{d} V(k, n)$.) In this sense, therefore, solving only homogeneous forms sacrifices no generality. The major problem, however, is to discover methods for selecting solutions to Equation 5.2 given solutions to Equation 5.3. Hopefully, recursive means will present themselves, thus avoiding the obvious but inefficient method of choosing those solutions to Equation 5.3 for which $x_{n+1} = 1$.

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COLOPHON

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This original composition was set in the 12pt Palatino and Euler Virtual Math fonts of Hermann Zapf for an A4 formatted page, the standard for this printing.