A χ-DISTRIBUTION MODEL OF HAIL STORM DAMAGE

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Centre of
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Prologue

"Hail, Caesar!"

— Brutus

Steps of the Senate in Rome
Ides of March, 44 B.C.

Hail storm damage pattern

In this talk we address the pattern of damage, and investigate its properties, of a theoretical hail storm which travels linearly across the landscape at constant velocity and gathers in intensity before subsiding. We start by assuming a simpler model, that of a storm which does not move, restricted to having an uncorrelated bivariate normal distribution of damage.

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This model, expressed in the natural polar coordinates, leads to a 1-dimensional pattern of damage as a function of the marginal radial distance conforming to the 2-dimensional χ-distribution. We then extend the model to the traveling form, allowing further for a correlation of the variables, extending, as well, to the multi-dimensional case.

Crop insurance claims

Motivation springs from the real-world damage

to agricultural crops by hail storms, and the pursuant insurance claims. Such claims routinely refer to distance from the storm center, and are known to respond to countervailing influences. Storm damage occurs with greatest intensity at the center, tapering to insignificance at distance. However, the total of claims filed for damage at the center is small, and increases as more and more claimants reside at greater distances from the center.

Fraudulent claims

The U.S. Department of Agriculture

maintains a large crop insurance program, extending to the billions of dollars. Unfortunately, some claims are fraudulent, and frequently they are related — through groups of farmers who act in collusion, extending to conspiring agents, even insurance companies. Naturally it is desirable to contain this fraud, and part of the investigative toolkit is a good understanding of where actual storm damage has occurred, and to what extent.

A false assumption:
The lognormal distribution correctly describes storm damage.

The U.S.D.A. has been laboring under a false assumption

that the pattern of damage in a hail storm is distributed lognormally by distance from the center of the storm. The implication of this assumption is that the intensity of the storm at its center is zero, a conclusion not consistent with the facts.

An example of the false lognormal implication

Bivariate probability measure to lognormal —

Here is an example of a bivariate probability measure which induces a lognormal measure on $\mathbb{R}^+$. This example is generated by "reverse engineering" the lognormal density function to select a bivariate density which has the desired property. In each case the measure is that implied by the density.

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Let $f(r)$ be a bivariate density, where r is the radius from the origin in polar coordinates, independent of the angle θ. Then,

The induced bivariate distribution

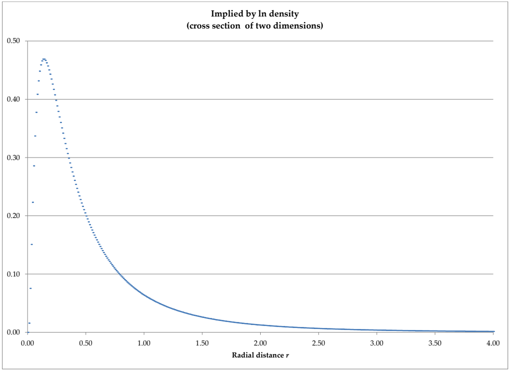
$$\int_0^{2\pi} rf(r)\,\mathrm{d}\theta = 2\pi rf(r)$$

$$= \frac{1}{r\sqrt{2\pi}} \exp\left(-\frac{\log^2 r}{2}\right),$$

the lognormal density. So

$$f(r) = \frac{1}{r^2(2\pi)^{3/2}} \exp\left(-\frac{\log^2 r}{2}\right)$$

The induced bivariate density (cross section)



Note: This function integrates to $\sqrt{\mathrm{e}}/(2\pi)$ in one dimension.

A proposal for storm damage, the χ-distribution

We propose an assumed hail storm damage pattern

which is bivariate normal. We advance the reason that a hail stone descending from a storm is subject to random buffeting much like that of other experiments which lead to the binomial distribution, *e.g.*, the Pascal triangle marble and peg demonstration, or the drunk stumbling away from a lamp post. The only difference here is that the randomness is in two dimensions.

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From this simple assumption comes the χ-distribution as the proper description of hail storm damage, measured by distance from the storm center.

The 2-dimensional χ-distribution
density and distribution

We choose polar coordinates as natural because we seek

the marginal distribution of the bivariate normal on $\mathbb{R}^+$ by integrating on the angular variable. The density and distribution then are

$$g(r) = \frac{1}{2\pi} r \exp\left(-\frac{r^2}{2}\right)$$

$$G(r) = \Pr\{R \leqslant r\} = \frac{1}{2\pi} \int_0^r s \exp\left(-\frac{s^2}{2}\right) \mathrm{d}s$$

$$= 1 - \exp\left(-\frac{r^2}{2}\right)$$

The multi-dimensional χ-distribution
density and distribution

We choose polar coordinates again as natural

because we seek the marginal distribution of the multi-variate normal on $\mathbb{R}^+$ by integrating all the angular variables. The density and distribution then are

$$g_d(r) := \frac{1}{\Gamma\left(\frac{d}{2}\right) \cdot 2^{(d/2)-1}} \cdot r^{d-1} \exp\left(-\frac{r^2}{2}\right)$$

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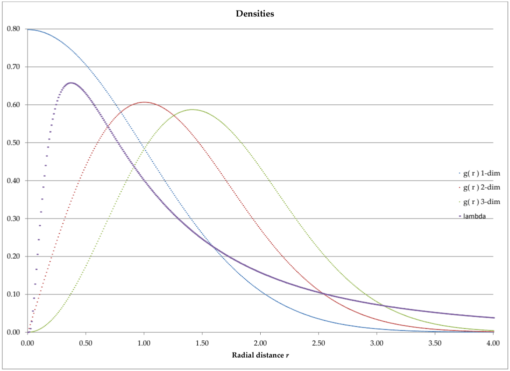
Multi-dimensional χ-densities
with lognormal



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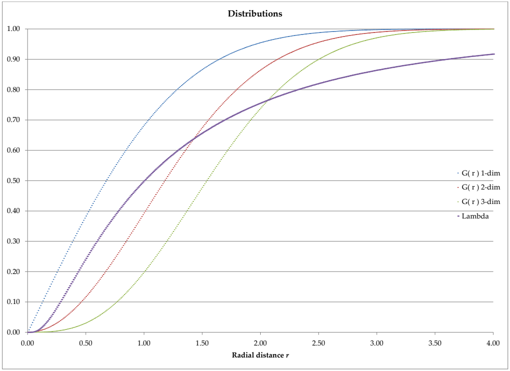
Multi-dimensional χ-distributions
with lognormal



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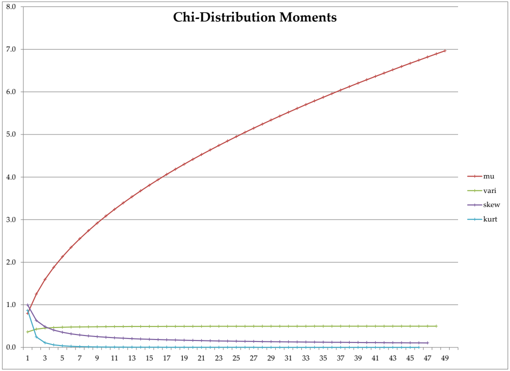
Multi-dimensional χ-moments



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New assumptions: movement and life cycle

Now we introduce two additional elements to the hailstorm.

We allow for linear movement at constant velocity, and for growth and decay. For the latter element we define the *intensity* I as a function of time t, as follows.

$$I(t) = \frac{1}{\sqrt{2\pi}\sigma_I} \exp\left(\frac{-t^2}{2\sigma_I^2}\right)$$

Note that this function satisfies the equation

$$I'(t) + \frac{t}{\sigma_I^2} I(t) = 0$$

Ellipsoidal total storm damage

These concepts of storm motion,

along with growth and decay of intensity, lead to accumulated storm damage with bivariate normal pattern, with associated diagonalized covariance matrix having eigenvalues equal to the semi-major and -minor ellipse axes.

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A generalization to multiple dimensions is forthcoming, with a *distance* concept to the level hyperellipsoids given by

$$R^2 = \frac{y_1^2}{a_1^2} + \cdots + \frac{y_n^2}{a_n^2},$$

where the $\{y_i\}$ and $\{a_i\}$ are space variables and semi-axes.

The total storm damage construction

Under these assumptions, the *total damage* $T(x_1, x_2)$ at the point (x_1, x_2) is given by the marginal distribution

$$
\begin{aligned}
T(x_1, x_2) &= \int_{-\infty}^{\infty} I(t) D_{v_1 t, v_2 t}(x_1, x_2)\,\mathrm{d}t \\
&= \frac{1}{(2\pi)^{3/2}\sigma_I} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\left[t^2/\sigma_I^2 + (x_1 - v_1 t)^2 + (x_2 - v_2 t)^2\right]}\,\mathrm{d}t.
\end{aligned}
$$

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The multi-variate χ-distributed total storm damage

Marginalizing out the angular random variables, one is left with the marginal radial random variable R with marginal probability density function

$$
\begin{aligned}
T_R(r) &= \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} T_Z(r, \phi_1, \ldots, \phi_{n-2}, \phi_{n-1}) \mathrm{d}\phi_{n-1} \mathrm{d}\phi_{n-2} \cdots \mathrm{d}\phi_1 \\
&= \frac{2^{1-n/2}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} \exp\left\{-\frac{r^2}{2}\right\}
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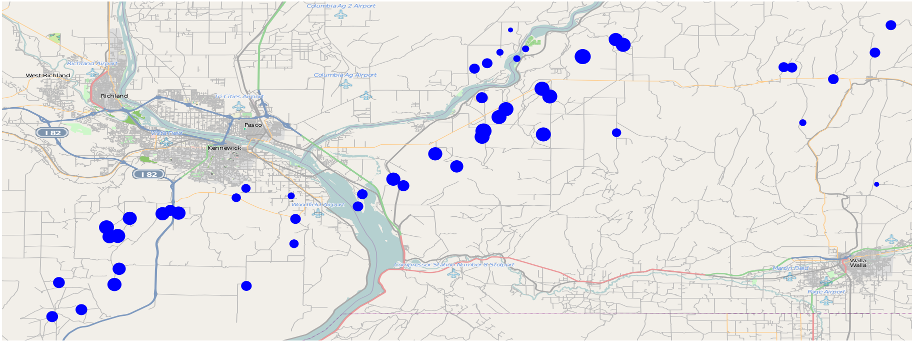
&= \frac{2^{1-n/2}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} \exp\left\{-\frac{r^2}{2}\right\}
\end{aligned}
$$

One recognizes $T_R(r)$ as the density function of the Chi distribution with n degrees of freedom. The interior of the ellipsoid has probability

$$
T_R(0 \leqslant R \leqslant r) = \int_0^r T_R(s) \mathrm{d}s = P(n/2, r^2/2),
$$

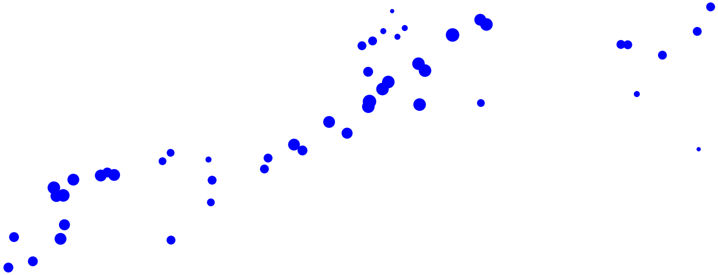
where P is the Regularized Gamma Function.

Hailstorm in the State of Washington

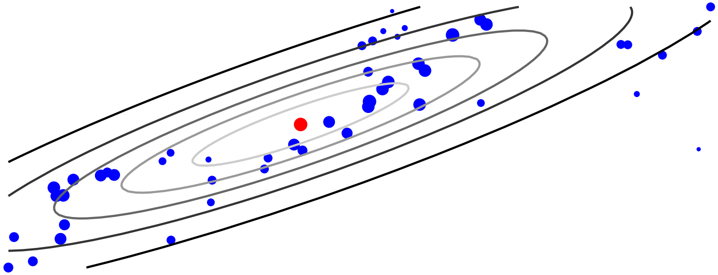


The dot sizes are proportional to the severity of the events.

Hail data

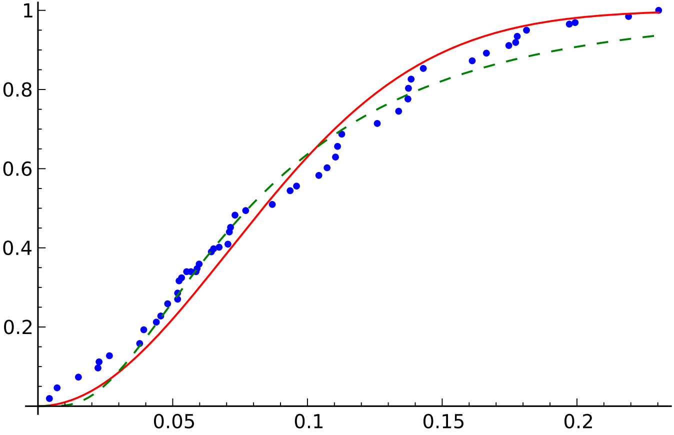


Hail data contours



These contours were fitted by the method of maximum likelihood.

Fitted χ-distribution and lognormal



These curves were fitted by a method of nonlinear least squares.

Final observations

Storm damage also depends on the geography

of the area considered. To test our model, we should use a data set over an area where the geography is homogeneous (*e.g.*, farm land or corn field agriculture) and the height is approximately constant. This problem is avoided in our case because our data are constructed from radar images.

After averaging, the damage is more regular than one would expect.

The Coriolis effect might give rise to curved data.

Future work

There are many stochastic properties of storms

and these are yet to be considered in our framework.

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As well, we recently have had the good fortune to receive copious amounts of data on which to test our theories. We will explore those with good vigor.

Epilogue

Not all those who wander are lost.

— J.R.R. Tolkien

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