HILBERT SPACE ANALYSIS
OF INTEGRAL EQUATIONS

PAUL C. KETTLER

ABSTRACT. This paper examines the theory of solving certain classes of integral equations using functional analysis methods of Hilbert Space. Included are equations with symmetric kernels and their associated self-adjoint linear operators, and in particular the Fredholm equation. The Hilbert Space methods, while taking time and effort to establish, prove rewarding in that they then simplify approaches to solving such equations, on the way providing more focused and powerful results. Examples serve to illustrate the principles.

1. INTRODUCTION TO THE TERMINOLOGY OF INTEGRAL EQUATIONS

A generalized linear integral equation is given by

$$a(x)u(x) + f(x) = \int_R K(x,t)u(t)\,dt$$

where $a(x)$, $f(x)$, and $K(x,t)$ are known and $u(x)$ is to be determined. R is a set with finite measure; the function $K(x,t)$ is called the *kernel*.

If $a(x) \equiv 0$, *i.e.*, coincides with the zero function almost everywhere, the equation is of the *first kind*.

If $a(x) \not\equiv 0$, the equation can be divided by it giving an equation of the *second kind* or *Fredholm type*.

If $a(x) \not\equiv 0$ and $f(x) \equiv 0$, the equation is homogeneous.

2. AN EXAMPLE OF AN ELEMENTARY INTEGRAL EQUATION SOLVED BY CLASSICAL METHODS

As a preliminary look into the nature of the integral equation consider the type most easily solved, those with *degenerate* kernels. A kernel is degenerate if it can be expressed as

$$K(x,t) = \sum_{i=1}^{n} a_i(x)b_i(t)$$

where a_i and b_i are functions defined on R.

Date: 3 September 2007.

2000 *Mathematics Subject Classification.* Primary: 45B05, 65R20. Secondary: 46E20, 47A50.

Key words and phrases. Hilbert Space, Symmetric kernels, Self-adjoint linear operators, Fredholm equation.

The author wishes to thank his adviser William Feller for valuable insights.

In August 2007 the author recomposed this paper from his original typescript of April 1962. He used the standard $\mathcal{AMS}$ article class and packages of the American Mathematical Society, and compiled the bibliography with BIBTEX, employing the University of Chicago Press style. He also included his own macros for specialized notation and convenience. This original composition was set in the 11pt Palatino and Euler Virtual Math fonts of Hermann Zapf for an A4 formatted page, the standard for this printing.

A Fredholm equation of this type is

$$u(x) = e^x + \lambda \int_0^{10} xtu(t)\,dt$$

where λ is a parameter.

$$K(x,t) = \lambda xt = (\lambda x)(t)$$

is a degenerate form.

Thus

$$u(x) = e^x + \lambda x \int_0^{10} tu(t)\,dt$$

Let

$$\int_0^{10} tu(t)\,dt =: C, \tag{2.1}$$

giving

$$u(x) = e^x + C\lambda x \tag{2.2}$$

Inserting Equation (2.2) into Equation (2.1)

$$\begin{aligned} C &= \int_0^{10} t(e^t + C\lambda t)\,dt \\ &= (t-1)e^t\Big]_0^{10} + C\lambda \frac{t^3}{3}\Big]_0^{10} \\ &= 9e^{10} - 1 + C\lambda\tfrac{1000}{3} \\ &= \frac{9e^{10}-1}{1-\lambda\frac{1000}{3}} \end{aligned}$$

$$\text{Therefore,}\quad u(x) = e^x + \frac{9e^{10}-1}{1-\lambda\frac{1000}{3}}\lambda x$$

In case $\lambda = \frac{3}{1000}$ the homogeneous equation

$$u(x) = \lambda \int_0^{10} xtu(t)\,dt$$

has solutions

$$u(x) = C\lambda x$$

where

$$C = C\lambda\tfrac{1000}{3}$$

by the foregoing reasoning with e^x and e^t removed, implying $C = C$, implying C is arbitrary.

Remark. If $K(x,t)$ has more than one term and $K(x,t)$ is degenerate a slightly more complicated solution evolves. More than one constant appears in the process of solution. Such constants will either have a linear relation (homogeneous case) or be uniquely determinable (non-homogeneous case.) An example is given in (Petrovskii 1957, pp. 15–17).

3. Two theorems relating the integral equation with symmetric kernel to a completely continuous self-adjoint linear operator in Hilbert Space

Although there are numerous different types of kernels to be considered in a systematic study of the integral equation, the symmetric kernel is of special interest because of the convenient relations to Hilbert Space. In the following discussion I will assume a knowledge of these definitions. For a general foundation in functional analysis one may read (Kolmogorov and Fomin 1957).

(1) linear operator
(2) completely continuous
(3) self adjoint
(4) complete
(5) orthogonal
(6) normal
(7) L^2, the space of square integrable functions
(8) orthogonal complement
(9) weak convergence
(10) strong convergence

Theorem 3.1. *Given*

(1) R *is a set with measure* μ,
(2) $K(x,t)$ *is a function defined on* R^2,
(3) $K(x,t) = K(t,x)$, *the symmetry condition,*
(4) $\int_{R^2} K^2(x,t)\,d\mu^2 < \infty$,

then the operator in L^2 *defined by the formula* $g(x) = \int_R K(x,t)u(t)\,dt$, *i.e.,* $g = Au$, *is completely continuous and self adjoint. In consequence, integral equations can then be treated with Hilbert Space methods.*

Proof. Let $\{\psi_n(x)\}$ be a complete orthonormal system in L^2. Then the set of all possible pairs $\psi_m(x)\psi_n(t)$ form a complete orthonormal system in R^2. Therefore, $K(x,t)$ can be represented as

$$K(x,t) = \sum_m \sum_n a_{mn}\psi_n(x)\psi_m(t),$$

where $a_{mn} = a_{nm}$ by symmetry. Also,

$$\int_{R^2} K^2(x,t)\,d\mu^2 = \sum_m \sum_n a_{mn}^2 < \infty \tag{3.1}$$

because of the orthonormality of the set $\psi_n(x)\psi_m(t)$.

Let

$$u(t) = \sum_n b_n \psi_n(t)$$

Then

$$\begin{aligned} g(s) &= \int_R K(s,t)u(t)\,dt \\ &= \int_R \left[\sum_m \sum_n a_{mn}\psi_n(s)\psi_m(t)\right]\left[\sum_n b_n\psi_n(t)\right] dt \\ &= \sum_n \left(\sum_m a_{mn} b_m\right)\psi_n(s) = \sum_n c_n \psi_n(s) \end{aligned}$$

and

$$\begin{aligned} c_n^2 &= \left(\sum_m a_{mn} b_m\right)^2 \leqslant \sum_m a_{mn}^2 \sum_m b_m^2 \\ &= \left(\sum_m a_{mn}^2\right)\|u\|^2 \end{aligned}$$

Now let

$$a_n := \sum_m a_{mn}^2$$

Since

$$\sum_{n=1}^{\infty} a_n = \sum_n \sum_m a_{mn}^2$$

converges by Equation (3.1), there exists an n_0 such that for any $\epsilon > 0$,

$$\sum_{n=n_0+1}^{\infty} a_n < \epsilon$$

This means that

$$\begin{aligned} \left\| g(s) - \sum_{n=1}^{n_0} c_n\psi_n(s)\right\|^2 &= \sum_{n=n_0+1}^{\infty} c_n^2 \quad \text{(by the completeness of } \{\psi_n\}) \\ &\leqslant \sum_{n=n_0+1}^{\infty} \|u\|^2 a_n^2 < \epsilon\,\|u\|^2 \end{aligned} \tag{3.2}$$

Let the sequence $\left\{u^{(k)}\right\}$ converge weakly to u. Then

$$u^{(k)}(t) = \sum_n b_n^{(k)} \psi_n(t) \longrightarrow u(t) = \sum_n b_n \psi_n(t)$$
$$\implies \left\{b_n^{(k)}\right\} \longrightarrow b_n$$

Since $\quad c_n^{(k)} = \sum_m a_{mn} b_m^{(k)}, \quad$ then

$$c_n^{(k)} \longrightarrow c_n, \quad \forall n$$

Therefore,

$$\sum_{n=1}^{n_0} c_n^{(k)} \psi_n(s) \longrightarrow \sum_{n=1}^{n_0} c_n \psi(s), \quad \text{for any } n_0 \tag{3.3}$$

Now $\left\|u^{(k)}\right\|$ is bounded. Therefore by Equation (3.2)

$$\left\| g^{(k)}(s) - \sum_{n=1}^{n_0} c_n^{(k)} \psi_n(s) \right\|^2 < \epsilon \left\| u^{(k)} \right\|^2,$$

$g^{(k)}(s)$ is arbitrarily close to $\sum_{n=1}^{n_0} c_n^{(k)} \psi_n(s)$ in norm, and this quantity converges in norm to $\sum_{n=1}^{n_0} c_n \psi_n(s)$ by Equation (3.3). Therefore $g^{(k)}(s) \to g(s)$ in norm, proving that the operator is completely continuous.

By Fubini's Theorem concerning the iteration of the Lebesgue integral

$$\begin{aligned}
(Au, g) &= \int_R \left(\int_R K(x,t) u(t) \, d\mu_t \right) g(x) \, d\mu_x \\
&= \int_R u(t) \left(\int_R K(x,t) g(x) \, d\mu_x \right) \, d\mu_t \\
&= (u, Ag),
\end{aligned}$$

proving that A is self adjoint. ☐

Theorem 3.2. *Given any completely continuous self-adjoint linear operator A on a Hilbert Space H, there exists an orthonormal system $\{\phi_n\}$ of eigenvectors corresponding to the eigenvalues $\{\lambda_n\}$ such that every element $\xi \in H$ may be written in a unique way as*

$$\xi \equiv \sum_k c_k \phi_k + \xi',$$

where ξ' satisfies the condition $A\xi' = 0$. In addition, $\lim_{n\to\infty} \lambda_n = 0$. (From the concept of eigenvector, eigenvalue, it is then known that $A\xi = \sum_k \lambda_k c_k \phi_k$.) As a consequence, an integral equation with symmetric kernel will have an eigenvalue expansion.

Proof. Deferred, for statement and proof of the following intermediate lemma —

Lemma 3.3. *Given that* $|Q(\xi)|$ *attains a maximum on the unit sphere at* ξ_0*, then*

$$(\xi_0, \eta) = 0 \implies (A\xi_0, \eta) = (\xi_0, A\eta) = 0$$

As a consequence, ξ_0 *is an eigenvector and* $\lambda_0 = (A\xi_0, \xi_0)$ *is the corresponding eigenvalue.*

Proof. The set of vectors $\{\eta\}$ constitute the orthogonal complement of the one-dimensional subspace spanned by $A\xi_0$. Therefore, $A\xi_0 = \lambda_0\xi_0$ and $(A\xi_0, \xi_0) = (\lambda_0\xi_0, \xi_0) = \lambda_0$. See (Kolmogorov and Fomin 1961, Corollary 1, p. 124), the statement of which is,

> The orthogonal complement to the orthogonal complement of a closed linear subspace M coincides with M itself.

□

Proof of Theorem 3.2. To prove that one eigenvector exists I examine the quadratic functional $Q(\xi) = (A\xi, \xi)$ for all points $\{\xi\}$ on the unit sphere in H. $(\xi, \xi) = 1$. It is necessary to show that $|Q(\xi)|$ attains a maximum on the unit sphere, and to find the point of such maximum.

Let $S_1 = \sup_{\|\xi\| \leqslant 1} |(A\xi, \xi)|$, *i.e.*, the supremum in the spherical ball. Then it is clear that S_1 occurs on the spherical surface because:

> Assume $S_1 = |(A\xi, \xi)|$, $\|\xi\| < 1$. Take the multiple of ξ of unit norm. Call this element $\lambda\xi$, $\lambda > 1$. Then $S_1' = |(A\lambda\xi, \lambda\xi)| = \lambda^2 |(A\xi, \xi)| > |(A\xi, \xi)|$, and the contradiction is apparent.

To find a point at which S_1 occurs, take a sequence of points $\{\xi_n\}$ such that

$$\|\xi_n\| \leqslant 1 \text{ and } |(A\xi_n, \xi_n)| \longrightarrow S_1 \tag{3.4}$$

Then using the fact that the unit sphere is weakly compact, a subsequence of $\{\xi_n\}$ can be selected such that $\{\xi_n\}$ converges weakly to some element η with

$$|(A\xi_n, \xi_n)| \longrightarrow |(A\eta, \eta)| = S_1,$$

following directly from Equation (3.4) and this result, (Kolmogorov and Fomin 1961, Lemma 1, p. 130), the statement of which is, with notation changes to conform to the present circumstances,

> If $\{\xi_n\}$ converges weakly to η and the self-adjoint linear operator A is completely continuous, then $(A\xi_n, \xi_n) \to (A\eta, \eta)$.

Therefore, one eigenvector, namely η, has been found. Take η for ϕ_1 to conform with the statement of the theorem. Then $A\phi_1 = \lambda_1\phi_1$, with

$$S_1 = |(A\phi_1, \phi_1)| = |\lambda_1(\phi_1, \phi_1)| = |\lambda_1|$$

By an inductive process assume that $\{\phi_1, \dots, \phi_n\}$ have been chosen with corresponding eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. Look at the functional $|(A\xi, \xi)|$ on the orthogonal complement of the space spanned by $\{\phi_1, \dots, \phi_n\}$. Then if $\alpha \in M(\phi_1, \dots, \phi_n) =: M$ and $\alpha' \in M_n'$, the orthogonal complement, $A\alpha \in M$ because all components are transformed into multiples of themselves. Therefore $(A\alpha, \alpha') = (\alpha, A\alpha') = 0$, *i.e.*, $A\alpha' \in M_n'$, and it is seen that M_n' is invariant under A.

Examine the unit sphere and $|(A\xi, \xi)| \in M_n'$. Choose an eigenvector ϕ_{n+1}. By the construction,

$$S_1 = |\lambda_1| \geqslant S_2 = |\lambda_2| \geqslant \cdots \geqslant S_n = |\lambda_n| \geqslant \cdots,$$

because $M_n' \supset M_m'$ for $m \geqslant n$.

Therefore, one of two cases may occur. Either (1) For some n_0, λ_n will be zero for $m > n_0$, or (2) $\lambda_n \neq 0$ for all n.

In Case (1), if $S_m = |\lambda_m| = 0$ for $m > n_0$, then $S_m = \sup_{\|\xi\| \leqslant 1} |(A\xi, \xi)| = 0$ implies $(A\xi, \xi) \equiv 0$ within the unit ball. Therefore, for any multiple of ξ outside the ball, *i.e.*, any vector in M_m', the same will be true. By Lemma 3.3, $A\xi_0 = (A\xi_0, \xi_0)\xi_0 = 0$ for all ξ_0 such that $\|\xi_0\| = 1$. For all multiples this will also be true, *i.e.*, A maps M_m' into zero.

In Case (2), to prove that $\{\lambda_m\} \to 0$ it is necessary to make use of the fact that every orthonormal system of vectors in H is weakly convergent to zero.

$$
\begin{aligned}
g_n &= (g, \phi_n) \quad \text{for any } g \in H \\
\sum g_n^2 &\leqslant \|g\| < \infty \quad \text{by Bessel's Inequality} \\
\text{Therefore,} \quad g_n &\longrightarrow 0 = (g, 0)
\end{aligned}
$$

Hence $\{A\phi_n\} = \{\lambda_n \phi_n\}$ converges to zero in norm because a completely continuous operator A transforms a weakly convergent sequence into a strongly convergent sequence. Thus $\lambda_n \phi_n \to 0$ implies $\lambda_n \to 0$ since $\|\phi_n\| = 1$. Now, if $M' = \cap_n M_n' \neq 0$ one must show that any vector $\xi \in M'$ is transformed to zero by A.

$|(A\xi, \xi)| \leqslant |\lambda_n| \, \|\xi\|^2$ for all n because:

> Let $\xi = \lambda_0 \xi_0$, for $\|\xi_0\| = 1$. Then $|(A\xi_0, \xi_0)| \leqslant |\lambda_n| \, (\xi_0, \xi_0)$ for all n (inasmuch as $\sup |(A\xi_0, \xi_0)| = |\lambda_n|$,) which implies
>
> $$
> \begin{aligned}
> |(A\lambda_0\xi_0, \lambda_0\xi_0)| &\leqslant |\lambda_n| \, (\lambda_0\xi_0, \lambda_0\xi_0) \\
> |(A\xi, \xi)| &\leqslant |\lambda_n| \, \|\xi\|^2 \longrightarrow 0 \\
> \textit{i.e.,} \quad (A\xi, \xi) &= 0
> \end{aligned}
> $$

By reasoning similar to that used in Case (1) above, $A\xi = 0$.

In conclusion it is seen that ξ can be represented as $\xi = \sum_k c_k \phi_k + \xi'$, where $A\xi' = 0$. Therefore, $A\xi = \sum_k \lambda_k c_k \phi_k$. In Case (1), the terms will all be zero after index $k = n_0$. In Case (2), the sum will be countably infinite. □

4. Hilbert Space solution of the Fredholm Equation

The Fredholm equation can be written

$$
u(x) = \int_R K(x, t) u(t) \, dt + f(x)
$$

Impose the symmetry condition $K(x,t) = K(t,x)$. Then the equation becomes a special case of the Hilbert Space equation $u = Au + f$. Let the functions $\{\phi_n\}$ defined on R be an orthonormal set corresponding to the non-zero eigenvalues. In Fredholm notation this means

$$\lambda_k \phi_k(x) = \int_R K(x,t)\phi_k(t)\,dt$$

Let $$u = \sum_n x_n \phi_n + u'$$

and $$f = \sum_n b_n \phi_n + f'$$

where $$Au' = Af' = 0$$

Then $$\sum_n x_n \phi_n + u' = \sum_n \lambda_n x_n \phi_n + \sum_n b_n \phi_n + f'$$

So $$\sum_n x_n(1-\lambda_n)\phi_n + u' = \sum_n b_n \phi_n + f'$$

The equation is satisfied if and only if

$$x_n(1-\lambda_n) = b_n$$

and $$u' = f'$$

i.e., if $$u' = f'$$

(4.1) and $$x_n = \frac{b_n}{1-\lambda_n} \quad \text{for } \lambda_n \neq 1$$

(4.2) $$b_n = 0 \quad \text{for } \lambda_n = 0$$

Equation (4.2) gives the condition for solutions to exist and Equation (4.1) gives the solutions.

5. Analogies between integral equation theorems and the more general Hilbert Space theorems

Several theorems in (Petrovskii 1957) are direct consequences of the theory developed above. I will prove several of these theorems in Hilbert Space terminology. For example, in §12 of Chapter III he proves the existence of eigenfunctions for integral equations with symmetric kernels by examining the integral form

(5.1) $$\int\int K(P,Q)\phi(P)\phi(Q)\,dP\,dQ$$

(5.2) If $$\phi(P) = \lambda \int K(P,Q)\phi(Q)\,dQ$$

as hypothesized in the theorem on page 57 of the cited reference, he then continues to prove that one finite eigenvalue λ exists. Equation (5.2) is just a special case of $\phi = \lambda A \phi$ for Hilbert Space. Therefore, Equation (5.1) is a special case of $(\phi, A\phi)$, the functional examined in the proof of Theorem 3.2. Petrovskii's proof proceeds in almost analogous fashion. Note that an eigenvalue in Petrovskii's terminology is the reciprocal of the eigenvalue in my terminology.

Theorem 3.2 proved that an orthonormal system of eigenvectors can be chosen (subject to the conditions of the theorem.) However, I have not yet shown that any two eigenfunctions, not necessarily chosen according to this special construction, are orthogonal.

Theorem 5.1. *Eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

Proof. Let

$$\begin{aligned} A\xi_1 &= \lambda_1\xi_1 \\ A\xi_2 &= \lambda_2\xi_2 \\ \lambda_1 &\neq \lambda_2 \\ \text{Then} \qquad (A\xi_1, \xi_2) &= \lambda_1(\xi_1, \xi_2) \\ (A\xi_2, \xi_1) &= \lambda_2(\xi_2, \xi_1) \\ (\lambda_1 - \lambda_2)(\xi_1, \xi_2) &= (A\xi_1, \xi_2) - (A\xi_2, \xi_1) \\ &= (A\xi_1, \xi_2) - (\xi_2, A\xi_1) = 0 \\ \text{Therefore,} \qquad (\xi_1, \xi_2) &= 0 \end{aligned}$$

□

Remark. For integral equations the terminology in this proof would be

$$\int K(x,t)\xi_1(t)\,dt = \lambda_1\xi_1(x)$$

$$\int K(x,t)\xi_2(t)\,dt = \lambda_2\xi_2(x), \quad \textit{etc.}$$

Theorem 5.2. *Eigenvalues of an equation with a symmetric kernel are all real.*

Proof. Assume functions defined on the field of complex numbers. Let

$$\begin{aligned} \phi &= \xi + i\xi' \\ \text{Then} \qquad \overline{\phi} &= \xi - i\xi' \\ \text{If} \qquad A\phi &= (a+ib)\phi \\ A(\xi + i\xi') &= (a+ib)(\xi + i\xi') \\ &= (a\xi - b\xi') + i(b\xi + a\xi') \\ \text{then} \quad A(\xi - i\xi') &= (a\xi - b\xi') - i(b\xi + a\xi') \\ &= (a-ib)(\xi - i\xi') \end{aligned}$$

Assuming $b \neq 0$, these two functions have distinct eigenvalues. Therefore, by Theorem 5.1,

$$\begin{aligned} (\xi + i\xi', \xi - i\xi') &= 0 \\ (\xi, \xi) + (\xi', \xi') &= 0 \\ \implies \quad \xi &\equiv \xi' \equiv 0 \\ \implies \quad \phi &\equiv 0, \end{aligned}$$

i.e., ϕ is not general, as assumed. Thus $(a+ib)$ with $b \neq 0$ cannot be an eigenvalue. □

Petrovskii has a theorem that eigenvectors corresponding to the *same* eigenvalue can be orthogonalized. His construction is essentially the Gram-Schmidt orthogonalization process, and his conclusion is that any system of eigenvectors can be chosen mutually orthogonal, a direct consequence of his theorem and Theorem 5.1, and a conclusion identical to the one reached in Theorem 3.2 with the construction using orthogonal complements.

To continue, assuming the following sequences of orthonormal eigenvectors and the corresponding eigenvalues

$$\phi_1(P), \phi_2(P), \phi_3(P), \ldots \tag{5.3}$$

$$\lambda_1, \lambda_2, \lambda_3, \ldots \tag{5.4}$$

Petrovskii then formulates a theorem, as here, with notation changes to conform to the present circumstances (Petrovskii 1957, Theorem 4, pp. 64–66).

Theorem 5.3 (Petrovskii)**.** *Let* $\phi_1(P)$ *be the eigenfunction of* $\phi(P) = \lambda \int K(P,Q)\phi(Q)\,dQ$ *belonging to the eigenvalue* λ_1*. Then one obtains for the kernel*

$$K_1(P,Q) = K(P,Q) - \frac{\phi_1(P)\phi_1(Q)}{\lambda_1}$$

the sequences of eigenfunctions and eigenvalues, analogous to the sequences (5.3) *and* (5.4) *for the kernel* $K(P,Q)$*, by deleting* $\phi_1(P)$ *and* λ_1*.*

Remark. I will prove the theorem using the Hilbert Space theory developed in preceding theorems and show the correspondences between the integral forms and the Hilbert Space elements and operations.

Proof. Multiply $K_1(P,Q)$ by $\phi(Q)$ and integrate with respect to Q.

$$\int K_1(P,Q)\phi(Q)\,dQ = \int K(P,Q)\phi(Q)\,dQ - \int \frac{\phi_1(P)\phi_1(Q)\phi(Q)}{\lambda_1}\,dQ$$

This equation corresponds to

$$A'\xi = A\xi - \lambda_1\xi_1(\xi_1.\xi) \tag{5.5}$$

(remembering that Petrovskii's eigenvalue notation gives reciprocals of the conventional Hilbert Space notation.)

Now, ξ_1 satisfies $\lambda_1\xi_1 = A\xi_1$. Therefore,

$$
\begin{array}{lrl}
\text{if} & A'\xi & = \lambda\xi \\
 & & = A\xi - \lambda_1\xi_1(\xi_1,\xi), \\
\text{then} & (\lambda_1\xi_1,\xi_1) & = (A\xi,\xi_1) - \lambda_1(\xi_1,\xi)(\xi_1,\xi_1) \\
 & & = (\xi, A\xi_1) - \lambda_1(\xi_1,\xi) \\
 & & = \lambda_1(\xi,\xi_1) - \lambda_1(\xi_1,\xi) = 0 \\
\text{So, either} & (\xi,\xi_1) = 0, & \\
\text{in which case} & \lambda\xi = A\xi, & \\
\text{or else} & (\xi,\xi_1) \neq 0 \implies \lambda = 0, &
\end{array}
$$

in which case ξ has a non-zero component which is a multiple of ξ_1. Inasmuch as A is invariant on the space spanned by ξ_1 and the orthogonal complement of this space, λ_1 must be zero, again implying $\lambda\xi = A\xi$. Conclusion: ξ, which is an eigenvector of A' with eigenvalue λ, is also an eigenvector of A with the same eigenvalue.

Conversely, for $i \geqslant 2$, $\phi_i(P)$ and λ_i from sequences (5.3) and (5.4) ($\{\xi_i\}$ and $\{\lambda_i\}$) are also eigenvectors and eigenvalues of $K_1(P, Q)$ or A'.

$$\begin{aligned} \lambda_i \xi_i &= A\xi_i \\ \text{and} \qquad (\xi_1, \xi_i) &= 0 \\ \text{Therefore,} \qquad \lambda_i \xi_i &= A'\xi_i + \lambda_1 \xi_1 (\xi_1, \xi_i) \\ &= A'\xi_i \end{aligned}$$

□

Remark. $\phi_1(Q)$ is not an eigenfunction of $K_1(P, Q)$, for if

$$\begin{aligned} \lambda_1 \xi_1 &= A\xi_1 - \lambda_1 \xi_1 (\xi_1, \xi_1) \\ \lambda_1 \xi_1 &= \lambda_1 \xi_1 - \lambda_1 \xi_1 (\xi_1, \xi_1) \\ \implies \lambda_1 &= 0 \end{aligned}$$

If Theorem 5.3 is applied successively to the kernels

$$K(P, Q), K_1(P, Q), K_2(P, Q), \ldots$$

i.e., analogous to the operators

$$A', A'', A''', \ldots, A^{(m)}, \ldots$$

where $A^{(m)}$ by induction becomes

$$A^{(m)} = A\xi - \sum_{k=1}^{m} \lambda_i \xi_i (\xi_i, \xi),$$

then the eigenvectors and eigenvalues of A for $i > m$ are eigenvectors and eigenvalues for $A^{(m)}$, and conversely.

Now, assuming Petrovskii's theorem proving the existence of one non-zero eigenvalue for an equation with non-vanishing kernel,[1] it is possible by using his "method of variations" to construct a sequence of eigenvectors for $K(P, Q)$ by examining successively the kernels

$$K(P, Q), K_1(P, Q), K_2(P, Q), \ldots$$

If in this process one chooses the eigenvectors to be an orthonormal set, this construction is directly analogous to the construction used in Theorem 3.2 in Hilbert Space notation. It is an inductive construction using orthogonal complements.

If there exist only a finite number of positive eigenvalues of A, then for some m, $A^{(m)}$ will have only zero eigenvalues. By Theorem 3.2, $A^{(m)}\xi = 0$ for all ξ. Therefore,

$$\begin{aligned} A^{(m)}\xi &= 0 \\ &= A\xi - \sum_{k=1}^{m} \lambda_i \xi_i (\xi_i, \xi) \\ \text{So} \quad A\xi &= \sum_{k=1}^{m} \lambda_i \xi_i (\xi_i, \xi) \end{aligned}$$

[1] If $K(P, Q) \equiv 0$, $\lambda = 0$ is an eigenvalue for any function.

This is a finite eigenvalue expansion of a symmetric kernel. Hence the kernel is *degenerate* according to definition. This is the same expression that appears in the statement of Theorem 3.2 where $c_i = (\xi_i, \xi)$, the Fourier coefficients.

The Hilbert-Schmidt Theorem and the Schmidt formula for the solution of an integral equation with symmetric kernel are just special cases of Theorem 3.2. The Hilbert-Schmidt Theorem states that if $f(P) = \int K(P,Q)h(Q)\,dQ$, where $h(Q)$ is a square integrable function, then $f(P)$ can be expanded in an absolutely and uniformly convergent series of eigenfunctions of the symmetric kernel $K(P,Q)$, *i.e.*, if

$$\begin{aligned} y &= Ax \\ \text{then} \quad y &= \sum_i \lambda_i x_i \phi_i \\ \text{where } x &= \sum_i x_i \phi_i \end{aligned}$$

and the $\{\phi_i\}$ are chosen orthonormal eigenfunctions of A. The Schmidt formula solves

$$\begin{aligned} \phi(P) &= \lambda \int K(P,Q)\phi(Q)\,dQ + f(P) \\ \text{or} \quad \phi &= \lambda A \phi + f \end{aligned}$$

in the same fashion as in Section 4 above.

REFERENCES

Kolmogorov, A. N. and S. V. Fomin (1957). *Elements of the Theory of Functions and Functional Analysis*, Volume I: Metric and Normed Spaces. New York: Graylock. Translated from the Russian by Leo F. Boron.

Kolmogorov, A. N. and S. V. Fomin (1961). *Elements of the Theory of Functions and Functional Analysis*, Volume II: Measure, Lebesgue Integrals, and Hilbert Space. New York: Academic. Translated from the Russian by Natascha Artin Brunswick and Alan Jeffrey.

Petrovskii, I. G. (1957). *Lectures on the Theory of Integral Equations*. Rochester: Graylock. Translated from the second revised (1951) Russian edition by Hyman Kamel and Horace Komm.

(Paul C. Kettler)
DEPARTMENT OF MATHEMATICS
FINE HALL
PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY 08540