Fréchet – Hoeffding lower limit copulas in higher dimensions

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History

Prologue

Investigators have incorporated copula theories into their studies of multivariate dependency phenomena for many years. Copulas in general, which include the basic probability version as well as the Lévy and utility varieties, are enjoying a surge of popularity with applications to economics and finance.

Ordinary copulas have a natural upper bound in all dimensions, the Fréchet – Hoeffding limit, after the pioneering work of Wassily Hoeffding and, later, Maurice René Fréchet. Among the well-understood phenomena in the bivariate case is that a natural lower limit copula also exists.

The two variable case

In this two variable case,

on a domain of the unit square $[0,1]^2$, the upper and lower limit copulas take this form.

$$C_{\uparrow}(u,v) := \min(u,v) \qquad \text{for the upper}$$
$$C_{\downarrow}(u,v) := \max(u+v-1,0) \qquad \text{for the lower}$$

The natural extension to a higher dimensions n for the upper limit copula is this.

$$C_{\uparrow}(u_1,u_2,\ldots,u_n) := \min(u_1,u_2,\ldots,u_n)$$

This function has all the necessary properties of a copula.

An ‘extension’ that did not work

An extension of the bivariate lower limit copula, however,

to the multidimensional case has not been forthcoming. One proposed extension takes the form

$$\widetilde{C}\downarrow(u_1,u_2,\ldots,u_n) := \max\left(u_1+u_2+\cdots+u_n-(n-1),0\right)$$

This function certainly has the necessary range of $[0,1]$, but is not a copula in dimensions $n>2$ because it does not have the n-increasing property. See (Nelsen 1998, Subsection 2.10 and Exercise 2.35) and (Fantazzini 2004, Subsection 2.1).

A graphical interpretation

Look now to the following two figures

which illustrate probability measures leading to the Fréchet – Hoeffding upper and lower limit copulas. These probability measures are uniform in two dimensions, concentrated, respectively, on the line segments for which the joint identity random variable (X, Y) has the relationships $X = Y$ and $X + Y = 1$.

Shown also are sample domains of integration of the respective distribution functions.

Figure 1

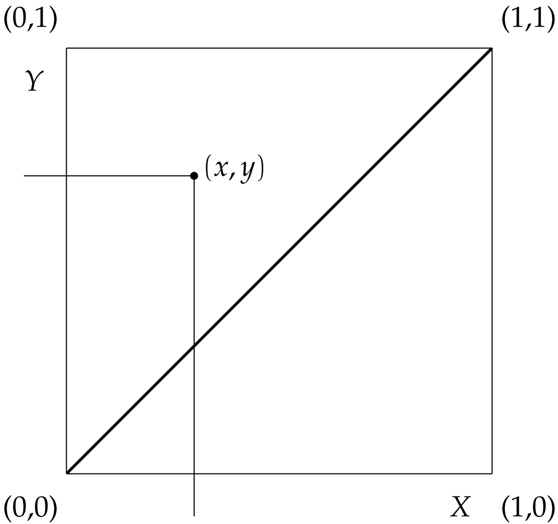

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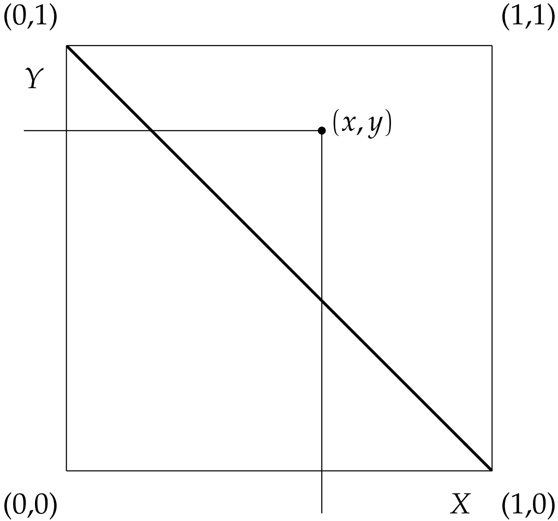
Figure 2



Other random variables have the same copulas.

Other relationships between random variables

also produce these copulas. In particular the relationship $Y = -X$ produces the second.

Perhaps this relationship caused the initial misunderstanding that the Fréchet – Hoeffding lower limit copula could not be extended beyond two dimensions. The false reasoning would go in the direction that multiple variables could not all have inverse binary relationships without all being zero.

A closer look at the figures

Observe carefully that the domain of concentration

in Figure 1 is referenced as the 'diagonal,' whereas the domain of concentration in Figure 2 is referenced as the 'simplex.' These distinctions are important, for in higher dimensions the corresponding domains continue to have the character of 'diagonal' and 'simplex.'

The copulas displayed above devolve from the following relationships (among others) of n random variables on unit hypercubes $[0,1]^n$.

$$X_1 = X_2 = \cdots = X_n$$
$$X_1 + X_2 + \cdots + X_n = 1$$

A three dimensional distribution extension —

For simplicity, let

$$\xi_i = 1 - x_i, \qquad i = 1,2,3$$
$$\xi_{ij} = \max\left(1 - (x_i + x_j), 0\right), \qquad (i,j) = (1,2),(1,3),(2,3)$$

Then

$$F(x_1, x_2, x_3) = \max\left(1 - (\xi_1^2 + \xi_2^2 + \xi_3^2), 0\right) + (\xi_{12}^2 + \xi_{13}^2 + \xi_{23}^2)$$

From this case the multidimensional form becomes evident.

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Realization of the multidimensional copula

The proposed definition

of the multidimensional lower limit copula is in fact the copula of the distribution function exhibited above for the multivariate identity random variable $(X_1, X_2, \ldots, X_n)$ with probability measure uniformly concentrated on the simplex.

A general distribution extension —

This study proposes an extension

of the Fréchet – Hoeffding lower limit distribution function and its copula. First, note the distribution function.

$$F_{\downarrow}(x_1, x_2, \ldots, x_n) \quad := \quad \max\left(1 - (\xi_1^{n-1} + \xi_2^{n-1} + \ldots + \xi_n^{n-1}), 0\right) + \sum_{i_1 < i_2}^{n} \xi_{i_1 i_2}^{n-1} + \ldots + (-1)^{n-1} \sum_{i_1 < i_2 < \ldots < i_{n-1}}^{n} \xi_{i_1 i_2 \cdots i_{n-1}}^{n-1},$$

where

$$\xi_{i_1 i_2 \ldots i_{k-1}} = \max\left(1 - \sum_{j=1}^{k-1} x_{i_j}, 0\right)$$

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The copula extension —

Next, note the copula extension,

first expressed parametrically, then non-parametrically.

$$C_{\downarrow}\left[1-(1-x_1)^{n-1}, 1-(1-x_2)^{n-1}, \ldots, 1-(1-x_n)^{n-1}\right] := F_{\downarrow}(x_1, x_2, \ldots, x_n),$$

so

$$C_{\downarrow}(u_1, u_2, \ldots, u_n) = F_{\downarrow}\left[1-(1-u_1)^{\frac{1}{n-1}}, 1-(1-u_2)^{\frac{1}{n-1}}, \ldots, 1-(1-u_n)^{\frac{1}{n-1}}\right]$$

The definition forms the basis of the developing study in all dimensions $n \geqslant 2$, extending the definition in dimension 2.

Definitions

The random variable terms of the simplex

of the Equation above shall be said to exhibit *complementary dependence*. The 2-dimensional relationship (with a 1-dimensional simplex) shall be said equivalently to exhibit *inverse dependence*.

The marginal distributions

Necessary to developing the copula

is the specification the marginal distributions. In the two dimensional case these were uniform on the unit interval, but in higher dimensions the corresponding measures concentrate increasingly toward the origin. This fact makes it necessary first to calculate the multivariate distribution function, and then its margins.

The higher-dimensional marginal measures —

To specify the marginal distributions

one need only look to the projections onto the axes of the uniformly concentrated measure on the $(n-1)$-simplex.

$$\mu_n(x_i) := (n-1)(1-x_i)^{n-2}, \text{ for } i = 1, 2, \ldots, n$$

The higher-dimensional marginal distributions —

These marginal measures integrate to

$$F_{\downarrow i} := 1 - (1 - x_i)^{n-1}$$

for the marginal distribution functions. Hence one has the arguments $\{F_{\downarrow i}\}$ of the Equation above for the parametric form of the extended Fréchet – Hoeffding lower bound copula.

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Epilogue

An attempt at visualizing the Fourth Dimension: Take a point, stretch it into a line, curl it into a circle, twist it into a sphere, and punch through the sphere.

— Albert Einstein

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