FRÉCHET – HOEFFDING LOWER LIMIT COPULAS IN HIGHER DIMENSIONS

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ABSTRACT. Investigators have incorporated copula theories into their studies of multivariate dependency phenomena for many years. Copulas in general, which include the basic probability version as well as the Lévy and utility varieties, are enjoying a surge of popularity with applications to economics and finance. Ordinary copulas have a natural upper bound in all dimensions, the so-called Fréchet – Hoeffding limit, after the pioneering work of Wassily Hoeffding and, later, Maurice René Fréchet, working independently. Among the well-understood phenomena in the bivariate case is that a natural lower limit copula also exists. An extension of this copula, however, to the multidimensional case has not been forthcoming. This paper proposes such an extension of the lower limit distribution function and its copula, and examines some of their properties.

1. INTRODUCTION

Investigators have incorporated copula theories into their studies of multivariate dependency phenomena for many years. Copulas in general, which include the basic probability version as well as the Lévy and utility varieties, are enjoying a surge of popularity with applications to economics and finance. Ordinary copulas have a natural upper bound in all dimensions, the so-called Fréchet – Hoeffding limit, after the pioneering work of Wassily Hoeffding (Hoeffding 1940; Hoeffding 1941) and, later, Maurice René Fréchet (Fréchet 1951; Fréchet 1958), working independently. Among the well-understood phenomena in the bivariate case is that a natural lower limit copula also exists.

In this two variable case, on a domain of the unit square $[0,1]^2$, the upper and lower limit copulas take this form.

$$C_{\uparrow}(u,v) := \min(u,v) \qquad \text{for the upper} \tag{1.1}$$

$$C_{\downarrow}(u,v) := \max(u+v-1,0) \qquad \text{for the lower} \tag{1.2}$$

The natural extension to a higher dimensions n for the upper limit copula is this.

$$C_{\uparrow}(u_1,u_2,\dots,u_n) := \min(u_1,u_2,\dots,u_n) \tag{1.3}$$

This function has all the necessary properties of a copula, justifying its name.

An extension of the bivariate lower limit copula, however, to the multidimensional case has not been forthcoming. One proposed extension takes the form

$$\widetilde{C}\downarrow(u_1,u_2,\dots,u_n) := \max\big(u_1+u_2+\cdots+u_n-(n-1),0\big)$$

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This function certainly has the necessary range of $[0, 1]$, but is not a copula in dimensions $n > 2$ because it does not have the n-increasing property. See (Nelsen 1998, Subsection 2.10 and Exercise 2.35) and (Fantazzini 2004, Subsection 2.1).

This paper proposes an extension of the Fréchet – Hoeffding lower limit distribution function and its copula. The starting point is the specification of a distribution function which induces the desired copula.

First consider the two-dimensional case leading to the copula formulas of Equations (1.1) and (1.2). Look then to Figures 1 and 2 which illustrate probability measures leading to these upper and lower limit copulas. The measures are uniform in two dimensions and are concentrated, respectively, on the line segments for which the joint identity random variable (X, Y) has the relationships $X = Y$ and $X + Y = 1$. Shown also are sample domains of integration of the respective distribution functions. Other relationships between random variables also produce these copulas. In particular the relationship $Y = -X$ produces the second.[1]

Note that distribution functions such as these, which are defined on the unit square and have uniform margins, are their own copulas. If $F(x, y)$ is such a distribution function with $F_1(x) = x$ and $F_2(y) = y$ its margins, and if $C(\cdot, \cdot)$ is its copula, then

$$C\big(F_1(x), F_2(y)\big) = F(x, y) = C(x, y)$$

With this observation, a distribution function in two dimensions which induces the lower bound copula of Equation (1.2) is

$$F(x, y) = \max(x + y - 1, 0) \tag{1.4}$$

The concept extends readily to distributions on the unit hypercube in higher dimensions.

2. Development of the multidimensional lower limit copula

Also observe that the domain of concentration in Figure 1 is referenced as the 'diagonal,' whereas the domain of concentration in Figure 2 is referenced as the 'simplex.' These distinctions are important, for in higher dimensions the corresponding domains continue to have the character of 'diagonal' and 'simplex.' The copulas of Equations (1.3) and (2.5) devolve from the following relationships (among others) of n random variables on unit hypercubes $[0, 1]^n$.

$$\begin{aligned} X_1 &= X_2 = \cdots = X_n \\ X_1 + X_2 + \cdots + X_n &= 1 \end{aligned} \tag{2.1}$$

Specifying a distribution function to extend Equation (1.2) utilizes the geometric relationship between the equilateral simplex of Equation (2.1) to the domain of integration for a value — the lower orthant to a chosen point $(x_1, x_2, \ldots, x_n)$. The intersection of this lower orthant with the plane of the simplex is itself an equilateral simplex of the same dimension $(n - 1)$. The two simplexes relate to each other in inverted position, that is, the reflection of one in barycentric space is congruent with the other by translation only, not rotation. Writing the distribution function therefore reduces to determining the measure of the intersection of these simplexes in all possible geometric combinations.

[1]Perhaps this relationship caused the initial misunderstanding that the Fréchet – Hoeffding lower limit copula could not be extended beyond two dimensions. The false reasoning would go in the direction that multiple variables could not all have inverse binary relationships without all being zero.

This specified mass is embodied in the following formulas, which consider all material relationships. First look at the three dimensional case, whence the general form becomes evident. For simplicity, let

$$\xi_i = 1 - x_i, \qquad i = 1,2,3$$
$$\xi_{ij} = \max\big(1 - (x_i + x_j), 0\big), \qquad (i,j) = (1,2),(1,3),(2,3)$$

Then

$$F(x_1, x_2, x_3) = \max\big(1 - (\xi_1^2 + \xi_2^2 + \xi_3^2), 0\big) + (\xi_{12}^2 + \xi_{13}^2 + \xi_{23}^2) \tag{2.2}$$

The proposed definition of the multidimensional lower limit copula as in Equation (2.5) is in fact the copula of the distribution function exhibited by Equation (2.3) for the multivariate identity random variable $(X_1, X_2, \ldots, X_n)$ with probability measure uniformly concentrated on the simplex of Equation (2.1).

Here are these functions. The terms of the distribution function after the first reflect the reversals of double counting when summing the components of the probability measure. The copula is first expressed parametrically, then non-parametrically.

$$\begin{aligned} F_\downarrow(x_1, x_2, \ldots, x_n) \quad &:= \quad \max\big(1 - (\xi_1^{n-1} + \xi_2^{n-1} + \ldots + \xi_n^{n-1}), 0\big) \\ + \sum_{i_1<i_2}^{n} \xi_{i_1 i_2}^{n-1} \quad &+ \quad \cdots \quad + \quad (-1)^{n-1} \sum_{i_1<i_2<\ldots<i_{n-1}}^{n} \xi_{i_1 i_2 \cdots i_{n-1}}^{n-1}, \end{aligned} \tag{2.3}$$

where

$$\xi_{i_1 i_2 \cdots i_{k-1}} = \max\big(1 - \sum_{j=1}^{k-1} x_{i_j}, 0\big)$$

Then

$$C_\downarrow\left[1 - (1-x_1)^{n-1}, 1 - (1-x_2)^{n-1}, \ldots, 1 - (1-x_n)^{n-1}\right] := F_\downarrow(x_1, x_2, \ldots, x_n), \tag{2.4}$$

so

$$C_\downarrow(u_1, u_2, \ldots, u_n) = F_\downarrow\left[1 - (1-u_1)^{\frac{1}{n-1}}, 1 - (1-u_2)^{\frac{1}{n-1}}, \ldots, 1 - (1-u_n)^{\frac{1}{n-1}}\right] \tag{2.5}$$

The formulation of Equation (2.5) provides the basis of the developing study in all dimensions $n \geqslant 2$, extending the definition of Equation (1.2) in dimension 2.

Definition 2.1. The random variable terms of the simplex of Equation (2.1) shall be said to exhibit *complementary dependence*. The 2-dimensional relationship (with a 1-dimensional simplex) shall be said equivalently to exhibit *inverse dependence*.

Necessary to developing the copula is the specification the marginal distributions. In the two dimensional case these were uniform on the unit interval, but in higher dimensions the corresponding measures concentrate increasingly toward the origin. This fact makes it necessary first to calculate the multivariate distribution function, and then its margins.

To specify the marginal distributions one need only look to the projections onto the axes of the uniformly concentrated measure on the $(n-1)$-simplex.

$$\mu_n(x_i) := (n-1)(1-x_i)^{n-2}, \text{ for } i = 1, 2, \ldots, n$$

These integrate to

$$F_{\downarrow i} := 1 - (1 - x_i)^{n-1}$$

for the marginal distribution functions. Hence one has the arguments $\{F_{\downarrow i}\}$ of Equation (2.4) for the parametric form of the extended Fréchet – Hoeffding lower bound copula.

3. Graphics

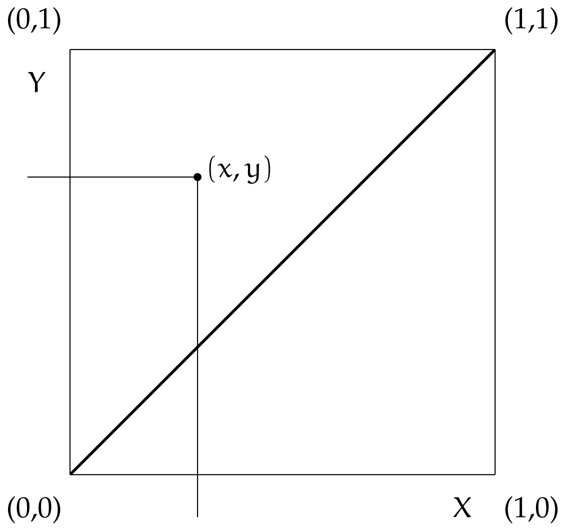


FIGURE 1. Mass concentration on diagonal $X = Y$, with sample domain of integration for the Fréchet – Hoeffding upper limit joint distribution function and copula $F_{\uparrow}(x, y) = C_{\uparrow}(x, y) = \min(x, y)$

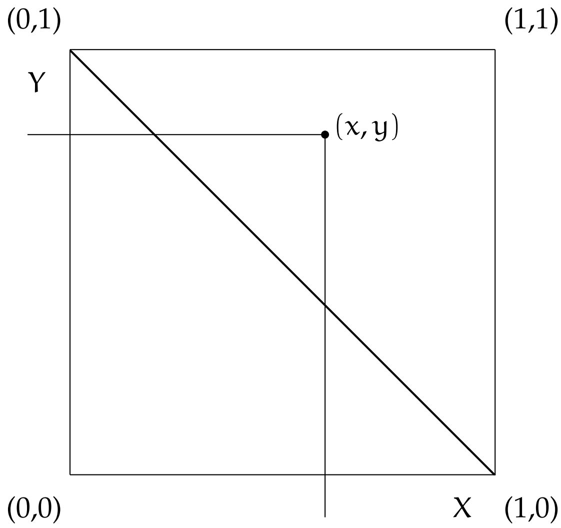
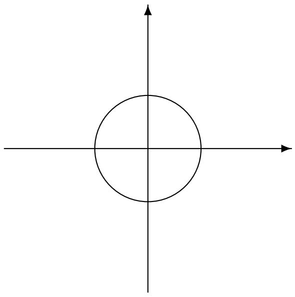
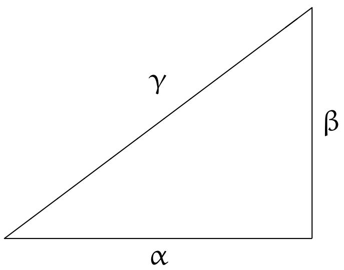
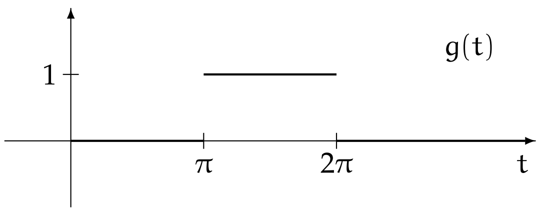


FIGURE 2. Mass concentration on simplex $X + Y = 1$, with sample domain of integration for Fréchet – Hoeffding lower limit joint distribution function and copula $F_{\downarrow}(x, y) = C_{\downarrow}(x, y) = \max(x + y - 1, 0)$




γ
β
α


g(t)
1
π
2π
t

References

Fantazzini, D. (2004, June). Copula's conditional dependence measures for portfolio management and Value at Risk. In L. Bauwers and W. Pohlmeier (Eds.), *Summer School in Economics and Econometrics of Market Microstructure*, Universität Konstanz.

Fréchet, M. R. (1951). Sur les tableaux de corrélation dont les marges sont données. *Ann. Univ. Lyon Sci. 4*, 53–84.

Fréchet, M. R. (1958). Remarques au sujet de la note précédente. *C. R. Acad. Sci. Paris Sér. I Math. 246*, 2719–2720.

Hoeffding, W. (1940). Scale-invariant correlation theory. In N. I. Fisher and P. K. Sen (Eds.), *The Collected Works of Wassily Hoeffding*, pp. 57–107. New York: Springer-Verlag.

Hoeffding, W. (1941). Scale-invariant correlation measures for discontinuous distributions. In N. I. Fisher and P. K. Sen (Eds.), *The Collected Works of Wassily Hoeffding*, pp. 109–133. New York: Springer-Verlag.

Nelsen, R. B. (1998). *An Introduction to Copulas*. New York: Springer-Verlag.

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