Sensitivity with respect to the yield curve

duration in a stochastic setting

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Prologue

"When I consider the short duration of my life, swallowed up in the eternity before and after, the little space which I fill and even can see, engulfed in the infinite immensity of spaces of which I am ignorant and which know me not, I am frightened and am astonished at being here rather than there; for there is no reason why here rather than there, why now rather than then. Who has put me here? By whose order and direction have this place and time been allotted to me?"

— Blaise Pascal, *Pensées*

Prologue (*en franąis*)

«Quand je considère la petite durée de ma vie absorbée dans l'éternité précédente et suivante, le petit espace que je remplis et même que je vois abîmé dans l'infinie immensité des espaces que j'ignore et qui m'ignorent, je m'effraye et m'étonne de me voir ici plutôt que là, car il n'y a point de raison pourquoi ici plutôt que là, pourquoi à présent plutôt que lors. Qui m'y a mis? Par l'ordre et la conduite de qui ce lieu et ce temps a-t-il été destiné à moi?»

— Blaise Pascal, Pensées

Duration in the financial marketplace

The bond market worldwide has about $85 trillion outstanding,

with about $1 trillion trading on a typical day. Other than price, the most widely quoted parameter in the market, without question, is duration. It appears on quotation screens, on traders' lips, and in all manner of literature on the market.

Yet the concept, which dates back 73 years, addresses the sensitivity of a bond's price with respect to changes in yield, assumes a uniform rate of interest through the life of a bond, an unrealistic posture.

Duration and convexity

In basic bond analysis one considers a zero coupon bond with present value v as a function of a level interest rate r, maturing to future value 1 at time T. The relationship of variables is this.

$$v = \mathrm{e}^{-rT}$$

The duration

$$d := \frac{1}{v}\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\log v = -T;$$

the convexity

$$c := \frac{1}{2v}\frac{\partial^2 v}{\partial r^2} = \frac{1}{2}T^2$$

Taylor series expansion

Note that d and c are the coefficients, respectively,

of r and r^2 in the Taylor series expansion of v.

$$v = 1 - rT + \frac{1}{2}r^2T^2 - \dots$$

Bond traders routinely employ duration and convexity in market analysis to estimate the effects of rate changes.

A piecewise constant rate

$$r(s) = \begin{cases} r_1 & \text{if} \quad 0 =: s_0 \leqslant s < s_1 \\ r_2 & \text{if} \quad s_1 \leqslant s < s_2 \\ \cdots & \\ r_n & \text{if} \quad s_{n-1} \leqslant s \leqslant s_n := T \end{cases}$$

$$v = \exp\left[-\sum_{i=1}^{n} r_i (s_i - s_{i-1}) \right]$$

Partial duration and convexity

From this expression the i^{th} *partial duration*

$$d_i := \frac{\partial}{\partial r_i} \log y = -(s_i - s_{i-1}), \qquad 1 \leqslant i \leqslant n$$

and the i^{th} *partial convexity*

$$c_i := \frac{1}{2}(s_i - s_{i-1})^2, \qquad 1 \leqslant i \leqslant n$$

Observe that the partial durations add to the total duration, whereas the partial convexities (and higher order related partial terms) do not.

Targeting duration

An active part of portfolio management is the targeting of a specific duration. For example, a pension fund manager may wish to have a value certain at some future time $t = T$, starting at $t = 0$ now.

Consider two portfolios A and B, with respective durations d_A and d_B, and present values (prices) of v_A and v_B. If these portfolios are combined, then the new portfolio $A + B$ has duration

$$d_{A+B} = \frac{v_A}{v_A + v_B} d_A + \frac{v_B}{v_A + v_B} d_B$$

Thus the duration of the combined portfolio is just the weighted average — by value — of the durations of the component parts, a very useful result.

Immunizing a portfolio

If A be the portfolio to be immunized to desired duration d_{A+B}, then one can solve for v_B knowing all other quantities. Specifically,

$$v_B = \frac{d_{A+B} - d_A}{d_B - d_{A+B}} \cdot v_A,$$

which may be positive or negative. If negative one can interpret the result as an amount proportioned to portfolio B to be sold from portfolio A to achieve the objective, or alternatively, the amount to sell short of portfolio B.

Big business

Bond immunization is a very big business.

In recent years Japanese banking interests have been heavy buyers of 30-year United States Treasury Bond strips — having a duration of 30 years — in order to extend the durations of portfolios. The activity has been so significant as to keep the longest-term yields below those of somewhat shorter-term yields for extended periods of time, even in strongly positive yield curve environments otherwise.

Varieties of yield curves

Yield curves come in two basic varieties,

the kinds which show

1. the instaneous rates of interest at various times in the future,
2. the average rates to the various future times.

Each type of curve is calculable from the other. Academic studies generally employ the former concept, sometimes called the "short rate," whereas business and commercial interests usually use the latter. Academicians ordinarily refer to risk-free zero-coupon bonds, whereas the marketplace may refer to coupon bonds of varying quality. In our paper, and in this talk, we choose the short rate, but first we show briefly the relationship between the two…

Shifting between the curves

Let $r(t)$ represent the short rate and $R(T)$ the average rate. Then,

$$R(T) = \frac{1}{T}\int_0^T r(t)\,\mathrm{d}t$$

by definition. So,

$$T \cdot R(T) = \int_0^T r(t)\,\mathrm{d}t$$

and therefore

$$T \cdot R'(T) + R(T) = r(T)$$

A more richly textured duration concept

Classical duration is just a number,

a point on the real line, like "10 years." One calculates it with reference to a level rate of interest throughout the life of a bond, or at least to a piecewise level rate. Such an assumption ignores the common sense that yield curves have shape, they are "curved," hence the name. A much more realistic point of view accepts this varied nature of future interest rates, and accommodates that view into analysis.

Now, therefore, we move forward not to consider a single interest rate changing through time, but rather an entire yield curve changing, and look to what implications that has for the response of bond prices.

A conceptual choice

We now ask ourselves,

"How do we map the evolution of the yield curve?" If today's curve goes from time 0 to time T does next year's go from time 1 to time $(T+1)$? Or should it also go from time 0 to time T? These are simply two different ways of looking at the same thing, but they are different.

If we wish to follow a specific bond which matures at time T, then we would prefer the former concept; we just look to time T, for example, time 10, on either curve, and read the rate. However, if we wish to follow a generic bond, always having a fixed maturity, like T, then we would prefer the latter concept.

Heracliitus and Parmenides

We have seen this choice before, in classical Greek philosophy,

with the impasse of Heraclitus and Parmenides, finally resolved by Plato with his theory of forms. Heraclitus was the "flux" man. When he saw a river he saw something different from the day before. Parmenides by contrast saw the same river each day, a constant. Plato's resolution was substantially in favor of Parmenides' reality, and is the view brought forward throughout Western philosophy today, with elements of Heraclitus' view yet remaining.

In fact, this ancient choice still presents itself in mathematics. For example, we have the "alias" view of a linear transformation — the axes move — and the "alibi" view — the points move.

The evolutionary yield curve

Herein we adopt the Parmenidean view,

that the yield curve is persistent, like the river. If we want to see the short rate for money 10 years into the future three years from now, we look to the value at 10 on that curve, not the value at 7, which would be the appropriate place were we to follow a specific 10-year bond, the Heraclitean view. To that end we now define a "yield surface" having two independent variables, one we call "space" and the other "time."

The yield surface

Insofar as we are preparing to follow

the evolving yield curve through time, we choose first the space variable, now symbolized by x, as the point of reference for any particular yield curve, at whatever time viewed in the future. Then we choose the time variable, now symbolized by t, as the epoch for a future entire yield curve. So, if we want the short rate alluded above we simply look to the yield surface at the point $(x, t) = (10, 3)$. The function defining this surface we now define as $f(x, t)$, or alternatively as $f_t(x)$.

The Musiela equation

We now state our dynamics for the evolutionary yield curve.

The transitional equation of state for the evolutionary yield curve is this:

$$\mathrm{d}f_t(x) = \frac{\mathrm{d}}{\mathrm{d}x} f_t(x) + \alpha_t(x)\,\mathrm{d}t + \sum_{k \geqslant 1} \sigma_t^{(k)}(x)\,\mathrm{d}B_t^{(k)}$$

The Musiela equation — related forms

Two modifications of the Musiela equation make economic sense. Changes are highlighted in red.

$$\mathrm{d}f_t(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} f_t(x)\right)^2 + \alpha_t(x)\,\mathrm{d}t + \sum_{k \geqslant 1} \sigma_t^{(k)}(x)\,\mathrm{d}B_t^{(k)}$$

$$\mathrm{d}f_t(x) = \left|\frac{\mathrm{d}}{\mathrm{d}x} f_t(x)\right| + \alpha_t(x)\,\mathrm{d}t + \sum_{k \geqslant 1} \sigma_t^{(k)}(x)\,\mathrm{d}B_t^{(k)}$$

The Malliavin derivative

The Malliavin derivative, of which several forms exist,

is the inverse operator of an appropriately defined stochastic integral, just as the ordinary [Newton] derivative is the inverse operator of an ordinary [Riemann or Lesbegue] integral.

Stochastic bond duration

With the Malliavin derivative in hand

we move forward to define the *stochastic duration*. Let F be a square integrable functional (providing the price relating to assumed cash flows of a specific bond) of the yield curve $\widehat{f}$ *wrt* $\widehat{P}$. Assume that F is Malliavin differentiable *wrt* $\widehat{f}$. Then the *stochastic duration of* F is stochastic process

$$D_{\cdot}F \in L^2(\Omega, \widehat{\mathfrak{F}}, \widehat{P}; K)$$

Herein, the variables circumflexed (hatted) are transformed versions of the original variables, made suitable for the standard analyses of the stochastic calculus.

Zero-coupon bond

$$P(t,T) = \exp\left(- \int_t^T f(t,s)\,\mathrm{d}s \right)$$

We find that

$$\begin{aligned} D_{r,y}\left(\int_0^{T-t} f_t(x)\,\mathrm{d}x \right) &= \int_0^{T-t} D_{r,y}\big(f_t(x)\big)\,\mathrm{d}x \\ &= \int_0^{T-t} \mathbf{1}_{[0,t]}(r)\,\mathrm{d}x \\ &= (T-t)\mathbf{1}_{[0,t]}(r) \end{aligned}$$

Interest-rate cap

$$
\begin{aligned}
D_{r,y}\left(\frac{1}{T-t}\int_t^T r(s)\,\mathrm{d}s\right) &= \frac{1}{T-t}\int_t^T D_{r,y}\big(r(s)\big)\,\mathrm{d}s \\
&= \frac{1}{T-t}\int_t^T D_{r,y}\big(f_s(0)\big)\,\mathrm{d}s \\
&= \mathbf{1}_{[0,t]}(r)
\end{aligned}
$$

Asian option

$$F = \frac{1}{(\overline{x}_2 - \overline{x}_1)(T_2 - T_1)} \int_{\overline{x}_1}^{\overline{x}_2} \int_{T_1}^{T_2} f_t(x) \, \mathrm{d}t \, \mathrm{d}x$$

Then

$$D_{r,y}F = \frac{1}{(\overline{x}_2 - \overline{x}_1)(T_2 - T_1)} \int_{\overline{x}_1}^{\overline{x}_2} \int_{T_1}^{T_2} \mathbf{1}_{[0,t]}(r) \, \mathrm{d}t \, \mathrm{d}x$$

$$= \mathbf{1}_{[0,t]}(r)$$

Stochastic duration in practice

Application ideas include these.

- Determining the yield curve maximum process
- Evaluating options dependent on the yield curve, *e.g.*, extended Black-Scholes
- Creating new derivative products based on functionals of the yield curve
- Implementing stochastic immunization strategies

Bond simulation with yield curve evolution

Utilizing a *backward* stochastic partial differential equation

(because the terminal valuation of a bond is known with certainty) one can develop numeric methods to track a bond's performance against an evolving yield curve, and accumulate data for statistical analyses.

The plan for the next paper in this series is to have extensive simulations and accompanying graphics.

Stochastic duration for better understanding

With the development of a stochastic duration concept we are now able to address problems heretofore unreachable. Any process which depends on an evolutionary yield curve is now suitable for investigation.

With future studies we will gain better understanding of what it means to react to a changing market because we will know more about that changing market itself. The result should be more effective decision making in all manner of activity where interest rates play a role.

Stochastic analysis terms from the paper

Absolutely continuous, Borel measurable, Brownian motion, BSDE, BSPDE, Chaos decomposition, FBSDE, FBSPDE, Filtration, Fréchet derivative, Functional, Galerkin approximation, Gateaux derivative, Gaussian random field, Girsanov's theorem, Hilbert space, Hilbert–Schmidt operator, Image, Indicator function, Integrating factor, Isometry, Kernel, Malliavin derivative, Martingale measure, Mild solution, Normal basis, Orthonormal basis, Predictable process, Probability space, Scalar product, Semigroup, Shift operator, Sobolov space, Stochastic differential equation, Strong solution, Tensor product, Wiener process

Epilogue

Not all those who wander are lost.

— J.R.R. Tolkien

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