Sensitivity with respect to the yield curve

duration in a stochastic setting

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Prologue

"When I consider the short duration of my life, swallowed up in the eternity before and after, the little space which I fill and even can see, engulfed in the infinite immensity of spaces of which I am ignorant and which know me not, I am frightened and am astonished at being here rather than there; for there is no reason why here rather than there, why now rather than then. Who has put me here? By whose order and direction have this place and time been allotted to me?"

— Blaise Pascal, *Pensées*

Prologue (*en français*)

«Quand je considère la petite durée de ma vie absorbée dans l'éternité précédente et suivante, le petit espace que je remplis et même que je vois abîmé dans l'infinie immensité des espaces que j'ignore et qui m'ignorent, je m'effraye et m'étonne de me voir ici plutôt que là, car il n'y a point de raison pourquoi ici plutôt que là, pourquoi à prÃľsent plutôt que lors. Qui m'y a mis? Par l'ordre et la conduite de qui ce lieu et ce temps a-t-il été destiné à moi?»

— Blaise Pascal, Pensées

Duration in the financial marketplace

The bond market worldwide has about $45 trillion outstanding,

with about $1 trillion trading on a typical day. Other than price, the most widely quoted parameter in the market, without question, is duration. It appears on quotation screens, on traders' lips, and in all manner of literature on the market.

Yet the concept, which dates back 70 years, addresses the sensitivity of a bond's price with respect to changes in yield, assumes a uniform rate of interest through the life of a bond, an unrealistic posture.

Duration and convexity

In basic bond analysis one considers a zero coupon bond with present value v as a function of a level interest rate r, maturing to future value 1 at time T. The relationship of variables is this.

$$v = \mathrm{e}^{-rT}$$

The duration

$$d := \frac{1}{v}\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\log v = -T;$$

the convexity

$$c := \frac{1}{2v}\frac{\partial^2 v}{\partial r^2} = \frac{1}{2}T^2$$

Taylor series comparison

Note that d and c are the coefficients, respectively,

of r and r^2 in the Taylor series expansion of v.

$$v = 1 - rT + \frac{1}{2}r^2T^2 - \ldots$$

Bond traders routinely employ duration and convexity in market analysis to estimate the effects of rate changes.

Portfolio duration

The duration of a portfolio is the average of the component durations weighted by present values. A portfolio of two bonds serves to illustrate. Let

$$v = \alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 \exp(-rT_1) + \alpha_2 \exp(-rT_2)$$

Then

$$d = -\frac{\alpha_1 v_1}{\alpha_1 v_1 + \alpha_2 v_2} T_1 - \frac{\alpha_2 v_2}{\alpha_1 v_1 + \alpha_2 v_2} T_2$$

A piecewise constant rate

$$r(s) = \begin{cases} r_1 & \text{if} \quad 0 =: s_0 \leqslant s < s_1 \\ r_2 & \text{if} \quad s_1 \leqslant s < s_2 \\ \cdots & \\ r_n & \text{if} \quad s_{n-1} \leqslant s \leqslant s_n := T \end{cases}$$

$$v = \exp\left[-\sum_{i=1}^{n} r_i (s_i - s_{i-1}) \right]$$

Partial duration and convexity

From this expression the i^{th} *partial duration*

$$d_i := \frac{\partial}{\partial r_i} \log y = -(s_i - s_{i-1}), \qquad 1 \leqslant i \leqslant n$$

and the i^{th} *partial convexity*

$$c_i := \frac{1}{2}(s_i - s_{i-1})^2, \qquad 1 \leqslant i \leqslant n$$

Observe that the partial durations add to the total duration, whereas the partial convexities (and higher order related partial terms) do not.

Stochastic paths for r and the $\{r_i\}$

Now restate v. $\quad v_t = \exp\left(-\int_t^T r(s)\,\mathrm{d}s\right), \quad 0 \leqslant t \leqslant T$

Replacing $r(s)$ with the process $R_t(s)$ defined on $[t, T+t]$ and evolving in a function space, v_t becomes the random variable V_t.

$$V_t = \exp\left(-\int_t^T R_t(s)\,\mathrm{d}s\right), \quad 0 \leqslant t \leqslant T$$

As anticipated, $\quad V_T = 1$.

The setup

Assume $r(s) \in U \subseteq C[0,T]$,

the Banach space of continuous functions on $[0,T]$ equipped with the *supremum* norm.

The set U is variously restricted by competing economic theories on the term structure of interest rates.

Pick a function $u(s) \in C[0,T]$ as a *direction*. Let $F\colon C[0,T] \mapsto \mathbb{R}$ be the Lebesgue integral.

Fréchet derivative

We state our first theorem,

with this definition provided in the proof.

Theorem: *The* generalized duration *of a bond is the negative of the integrated direction.*

Proof: Let $\vartheta := \int_0^T u(s)\,\mathrm{d}s$; so $|\vartheta| \leqslant \|u(s)\|_{L^\infty} T$. Then for $h > 0$, form this limit of difference quotients, the *directional derivative* of $F\big(r(s), u(s)\big)$.

(cont.)

Proof (cont.)

$$
\begin{aligned}
& \mathrm{d}F\big(r(s),u(s)\big) \\
:=& \lim_{h\to 0}\frac{1}{h}\left\{\exp\left[-\int_0^T \big(r(s)+hu(s)\big)\,\mathrm{d}s\right]-\exp\left[-\int_0^T r(s)\,\mathrm{d}s\right]\right\} \\
=& \lim_{h\to 0}\frac{1}{h}\left\{\exp\left[-\int_0^T r(s)\,\mathrm{d}s\right]\exp\left[-h\int_0^T u(s)\,\mathrm{d}s\right]-\exp\left[-\int_0^T r(s)\,\mathrm{d}s\right]\right\} \\
=& \lim_{h\to 0}\frac{1}{h}\left\{\exp\left[-\int_0^T r(s)\,\mathrm{d}s\right]\exp(-h\vartheta)-\exp\left[-\int_0^T r(s)\,\mathrm{d}s\right]\right\} \\
=& \lim_{h\to 0}\frac{1}{h}\left\{\exp\left[-\int_0^T r(s)\,\mathrm{d}s\right]\big(\exp(-h\vartheta)-1\big)\right\}
\end{aligned}
$$

(cont.)

Proof (cont.)

$$
\begin{aligned}
&= -\vartheta \exp\left[-\int_0^T r(s)\,\mathrm{d}s\right] \\
&= -\vartheta\, v\big(r(s)\big)
\end{aligned}
$$

The *generalized duration* of $v\big(r(s)\big)$

$$
\mathfrak{D}\big(r(s), u(s)\big) = \frac{1}{v\big(r(s)\big)}\,\mathrm{d}F\big(r(s), u(s)\big) = -\vartheta \qquad \square
$$

Remark

The function $F\big(r(s), u(s)\big)$ is a Fréchet derivative.

The essence of this Theorem is that it matters not how the yield curve is changing in shape insofar as duration is concerned.

What only matters is the integrated direction. In particular, duration is independent of the yield curve itself.

Corollaries

Corollary: If $\vartheta = T$,

then $\mathfrak{D}\big(r(s), u(s)\big) = -T$.

Corollary: If $u(s) \equiv 1$,

then $\mathfrak{D}\big(r(s), u(s)\big) = -T$.

If $r(s) \equiv r$, a constant, then this Corollary is a restatement of the classical duration.

$$d = \frac{1}{v}\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\log v = -T$$

Brownian motion exclusion

In framing the development we have excluded Brownian motion

for the reason that yields curves, in the absence of hyper-inflation or hyper-deflation, are constrained either by natural economic forces or by government intervention. Therefore, any processes which assign positive probabilities to unbounded sets of paths are ignored.

Domains

Consider two time intervals,

the first $I_1 = [t, T_1]$ corresponding to the remaining life of a zero coupon bond, and the second, $I_2 = [t, t + T_2]$, the interval for a yield curve. We require that $t \in I_1$, so that the bond is outstanding, and that $T \in I_2$, so that the remaining life of the bond is covered by the yield curve. Note that a sufficient condition is $T_1 \leqslant T_2$.

Paths

We consider a path of a stochastic yield curve $R_t(\omega_2, s)$

on $(\omega_2, s, t) \in \Omega_2 \times I_2 \times I_2$, with values in $C[I_2]$. The variable t fixes the point at which the yield curve is observed, and the variable s spans its domain. This process restricted to the domain of the chosen security $\Omega_1 \times I_1 \times I_1$ induces a path ω_1 in the truncated state space.

Processes

Consider again the deterministic equation

$$v_t = \exp\big(-\int_t^T r(s)\,\mathrm{d}s\big),$$

this time restating $r(s)$ as $R_t(s)$ as a stochastic process on $[t, T]$. The meaning here is that the function $R_t(s) \in C[I_2]$ is on a stochastic path in that space. In particular, it does not mean that $R_t(s)$ for fixed t is developed as a stochastic path over $s \in [t, T]$.

The latter interpretation would imply that the path $R_t(s)$ is not known at time $s = t$ (except for its initial point,) whereas the former interpretation implies that the entire curve is known then.

The Musiela equation

Assume the typical probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and look at this equation.

$$\mathrm{d}R_t(s) = \frac{\mathrm{d}}{\mathrm{d}s}R_t(s) + b_t(s) + R_t(s)\alpha_t(s)\,\mathrm{d}t + \sum_{k\geqslant 1}\delta_t^{(k)}(s)\,\mathrm{d}B_t^{(k)}$$

is the transitional equation of state we adopt for the evolutionary yield curve. This specification includes an Ornstein-Uhlenbeck-type process, and assumes a centered distribution, which can be shifted by Girsanov.

The Musiela equation — related forms

Two modifications of the Musiela equation make economic sense. Changes are highlighted.

$$\mathrm{d}R_t(s) = \left(\frac{\mathrm{d}}{\mathrm{d}s}R_t(s)\right)^2 + b_t(s) + R_t(s)\alpha_t(s)\,\mathrm{d}t + \sum_{k\geqslant 1}\delta_t^{(k)}(s)\,\mathrm{d}B_t^{(k)}$$

$$\mathrm{d}R_t(s) = \left|\frac{\mathrm{d}}{\mathrm{d}s}R_t(s)\right| + b_t(s) + R_t(s)\alpha_t(s)\,\mathrm{d}t + \sum_{k\geqslant 1}\delta_t^{(k)}(s)\,\mathrm{d}B_t^{(k)}$$

The Ogawa integral

We take the Ogawa integral, a symmetric integral,

as the integral of choice for the stochastic yield curve. Under mild conditions it is equivalent to the Skorokhod integral. This choice, which is applicable to non-adapted processes, is a generalization of the Itô integral, and enables the definition of the Malliavin derivative as its dual.

$$\delta^{O}(g) = \int_0^u g(t,s)\,\mathrm{d}R_t(s),$$

where $R_t(s)$ is a solution to the Musiela equation.

The Malliavin derivative

The Malliavin derivative $D_{t,s}F$ of a random variable F

is defined dual to the Ogawa integral, as follows. The definition is valid on the domain of variables for which the right hand side exists.

$$\mathrm{E}\left[\int_0^T\int_0^\infty (\mathrm{D}_{t,s}F)g(t,s)\,\mathrm{d}t\,\mathrm{d}s\right] = \mathrm{E}\left[F\int_0^T g(t,s)\,\mathrm{d}R_t(s)\right]$$

Sensitivity with respect to the yield curve

With the Malliavin derivative in hand

we move forward to define the *stochastic duration*.

$$\mathbb{D}_{t,s} V_t(s) := \mathrm{D}_{t,s} V_t(s),$$

where the latter is defined. Typically, an evaluation takes the form

$$\mathbb{D}_{t,s} V_t(s) = U_t(s) V_t(s)$$

In this form $U_t(s)$ is analogous to the original simple duration. The logarithm does not apply in this setting, so we make only passing reference to it now.

Stochastic bond duration

Now the presentation of *stochastic bond duration* is straightforward, going back to the formulation of V_t.

$$\mathbb{D}_{t,s} V_t = \mathbb{D}_{t,s} \exp\left(-\int_t^T R_t(s)\,\mathrm{d}s \right), \quad 0 \leqslant t \leqslant T$$
$$= \exp\left(-\int_t^T R_t(s)\,\mathrm{d}s \right)(t - T)$$

Stochastic duration in practice

Application ideas include these.

- Determining the yield curve maximum process
- Evaluating options dependent on the yield curve, *e.g.*, extended Black-Scholes
- Creating new derivative products based on functions of the yield curve.

Yield curve evolution

Utilizing a backward stochastic differential equation

(because the terminal valuation of a bond is known with certainty) one can develop numeric methods to track a yield curve and accumulate statistics.

Yield curve evolution

With the development of a stochastic duration concept we are now able to address problems heretofore unreachable. Any process which depends on an evolutionary yield curve, not just on the average value of a projected interest rate, is now suitable for investigation. As well, other related concepts such as stochastic convexity easily follow, opening an entire study on how changing yield curves in time and space influence all manner of financial variables.

Epilogue

Not all those who wander are lost.

— J.R.R. Tolkien

References, page 1

Brace, A., D. Gątarek, and M. Musiela (1997, April).
The market model of interest rate dynamics.
Math. Finance 7(2), 127–155.

Brace, A. and M. Musiela (1994, July).
A multifactor Gauss Markov implementation of Heath, Jarrow, and Morton.
Math. Finance 4(3), 259–283.

Gawarecki, L. P. (1999).
Transformations of index set for Skorokhod integral with respect to Gaussian processes.
J. Appl. Math. Stoch. Anal. 12(2), 105–111.

References, page 2

- Gawarecki, L. P. and V. S. Mandrekar (1993).
  Itô-Ramer, Skorohod and Ogawa integrals with respect to Gaussian processes and their interrelationship.
  In C. Houdré and V. Pérez-Abreu (Eds.), *Chaos Expansions, Multiple Wiener-Itô Integrals and Their Applications*, Probab. Stochastics Ser., Chapter 18, pp. 349–373. London: CRC Press.
- Macaulay, F. R. (1938).
  Some theoretical problems suggested by the movements of interest rates, bond yields and stock prices in the United States since 1856.
  New York: Columbia University Press.
- Vargiolu, T. (1999).
  Invariant measures for the Musiela equation with deterministic diffusion term.
  Finance Stoch. 3, 483–492.

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