Solution to "The '13' factor problem"

"What is the smallest natural number which, when multiplied by 13, gives a product of all 1's? What is the second smallest number? Provide a closed formula for the n^{th} such number."

The idea here is not to go looking for a number to multiply by 13, but rather to perform a long division of 13 into an endless string of 1's, until an exact quotient is obtained.

See the long division below. The quotient cycle is {0 0 8 5 4 7}, whereas the remainder cycle is {1 11 7 6 9 0}, as emphasized by italics. The two cycles iterate if and when a remainder of zero is achieved, as happens here at the sixth element. Thus 13 divides 111111 exactly 8547 times. As the zero remainder is first achieved following these digits, 8547 is the smallest multiple of 13 to provide a digit string of all 1's in the product. The next number in sequence giving the product of all 1's is 8547008547. In this event there are 12 straight 1's.

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    008547008547 … 008547 …
 13)111111111111 … 111111 …
    0
    11
     0
    111
    104
      71
      65
       61
       52
        91
        91
         0, etc.
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This problem has many algebraic ramifications. How long are the cycles? Are there any divisors which never produce a zero remainder? (Sure. Ten times any integer always ends in a zero, therefore not a 1.) What is the connection to the prime numbers? Can the cycle length be deduced without running the algorithm? (No. This is a fundamental result from the theory of algorithms, going back to the foundations of computing theory as devised by Alan Turing.) How are things different, or the same, when changing bases? like to 2? or in the abstract to e? What happens if one changes the dividend to all 2's, or to all 3's, etc.? (Not much, the quotients are simply 2 times, or 3 times, those shown for all 1's.)

The general term is $a_n = 8547 \cdot \frac{10^{6n}-1}{10^6-1}$. See your high school algebra text on geometric series. Note that the 100th number in this sequence, a_{100}, has 598 digits, a pretty big number, much larger than a googol, which only has 101 digits.